AMAT 583 Homework 5. Due Tuesday, December 3rd, 2019

1. Give an example of a set $S \subset \mathbb{R}$ which is homotopy equivalent but not homeomorphic to the 2-element set $\{0, 1\}$.

2. Give an example of a set $S \subset \mathbb{R}^3$ which is homotopy equivalent but not homeomorphic to the sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

3. Write down a deformation retraction from $I \times I$ to $I \times \{0\}$. (This is a map $h : I \times I \times I \to I \times I$ satisfying three properties. See the notes for Lecture 16-17.)

4. For each graph $G = (V, E)$, say whether G is a tree, a forest, or neither. (Recall: Working in the setting of undirected graphs, a forest is a graph with no cycles, and a tree is a forest with a single connected component.)

a. $V = \{a, b, c, d, e\}$, $E = \{[a, b], [a, c], [a, d], [a, e]\}$.
b. $V = \{a, b, c, d, e\}$, $E = \{[a, b], [b, c], [c, d]\}$,
c. $V = \{a, b, c, d, e, f\}$, $E = \{[a, b], [b, c], [c, a], [d, e], [e, f]\}$,
d. $V = \{a, b, c\}$, $E = \{\}$.
e. $G =$ the complete graph on 4 vertices.

5. Let

$$X = \{0, 1.5, 3, 12, 13, 14.5\} \subset \mathbb{R},$$

and let $Y = X \cup \{8\}$. Regard X and Y as metric spaces with the usual Euclidean metric d, i.e., $d(x, y) = |x - y|$. Plot Y, and compute and plot the (trimmed) single linkage dendrograms for X and Y. [Hint 1: One needs only to consider distances between consecutively ordered points. Hint 2: It may be helpful to use the algorithm outlined in class for computing single linkage dendrograms. While that algorithm was stated for the special case of $\mathbb{N}$-valued metrics, it in fact also works without modification for arbitrary metrics.]

Motivation/context for this exercise: In class, I have emphasized that the single-linkage dendrogram is not robust with respect to outliers. This exercise illustrates this idea in a very simple setting; here X has two clusters, each consisting of three points, sand Y contains these two clusters plus one outlier.

6. [Optional Bonus Exercise] Prove that the (untrimmed) dendrogram of a discrete hierarchical partition is a forest.