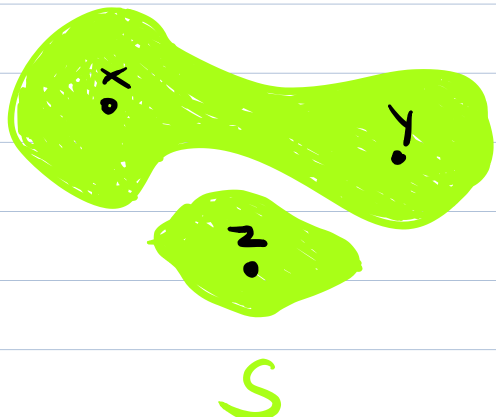
AMAT 583 Lec 10, 9/26/19

Today: Path components continued
Metric Spaces

Review: For $S \subset \mathbb{R}^n$, we defined an equivalence relation $\sim$ on S by taking $x \sim y$ if and only if $\exists$ a path from x to y.



$x \sim y$, but $x \not\sim z$ and $y \not\sim z$.

An equivalence class of $\sim$ is called a path component of S.

We denote the set of path components of S by $\pi(S)$.

Proposition: If $f: S \to T$ is a homeomorphism, then there is a bijection from $\pi(S)$ to $\pi(T)$.

Proof:

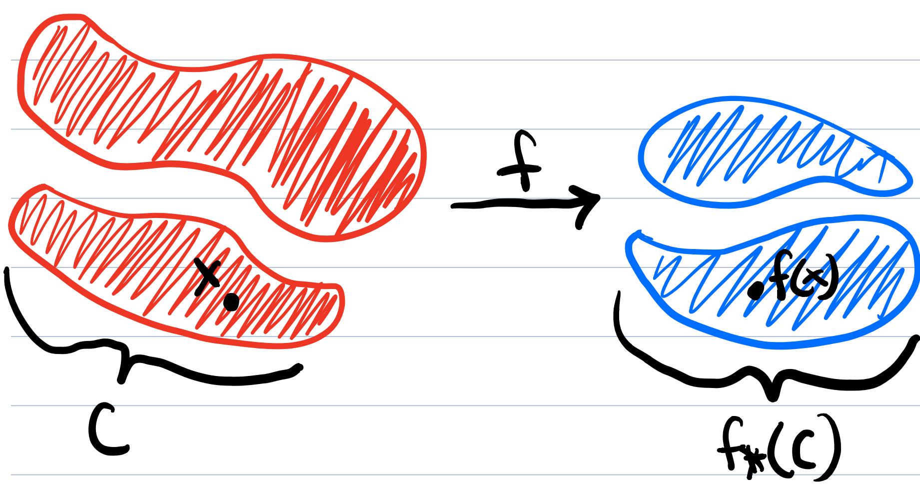
For $f: S \to T$ any continuous map, define a function

$$f_*: \pi(S) \to \pi(T) \text{ by the formula}$$

$$f_*([x]) = [f(x)].$$

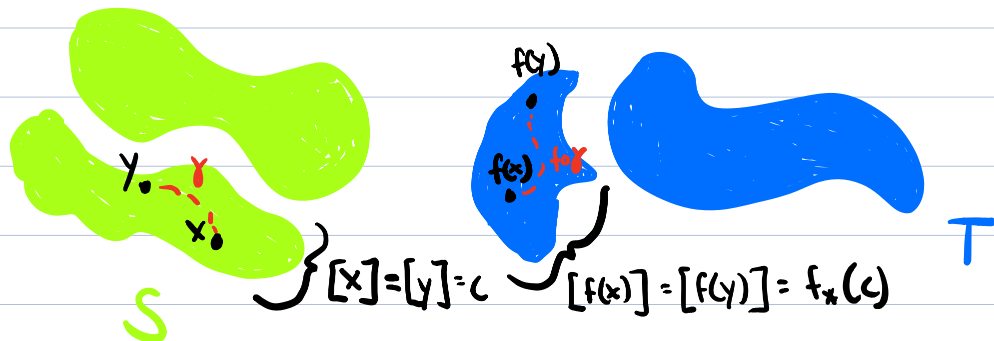
call this the map on path components induced by f

That is, if $C \in \pi(S)$, choose $x \in C$. Define $f_*(C) =$ the path component containing $f(x)$.



To show f_* is well defined we need to check that this definition doesn't depend on the choice of $x \in C$.

Illustration of the argument that f_* is well defined



That is, we need to check that if $[x]=[y]$ then $[f(x)]=[f(y)]$.

Here's the check:

If $[x]=[y]$, then $x \sim y$, i.e., there is a path $\gamma: I \to S$ from x to y. $f \circ \gamma : I \to T$ is a path from $f(x)$ to $f(y)$, so $f(x) \sim f(y)$, which implies $[f(x)]=[f(y)]$. ✓

The proof will rely on two basic facts about induced maps on path components:

1) For any $S \subset \mathbb{R}^n$, $Id^S_* = Id^{\pi(S)}$.

In words, the map on path components induced by the identity is the identity.

2) For any continuous maps $f: S \to T$, $g: T \to U$, $(g \circ f)_* = g_* \circ f_*$.

Proof: 1) $Id^S_* = Id^{\pi(S)} : \pi(S) \to \pi(S)$.

Pf: $Id^S_*([x]) = [Id(x)] = [x]$.

2) $(g\circ f)_*([x]) = [g\circ f(x)] = [g(f(x))] =$
$g_*([f(x)]) = g_*(f_*[x]) = g_*\circ f_*([x])$. $\blacksquare$

Assume $f: S \to T$ is a homeomorphism.
Then f, f^{-1} are both continous, and we have
$f^{-1}\circ f = Id^S$
$f\circ f^{-1} \; Id^T$

$$\text{Thus, } (f^{-1}\circ f)_* = \overset{Id^{\pi(S)}}{\overset{\|}{Id^S_*}} \Rightarrow f^{-1}_*\circ f_* = Id^{\pi(S)}$$

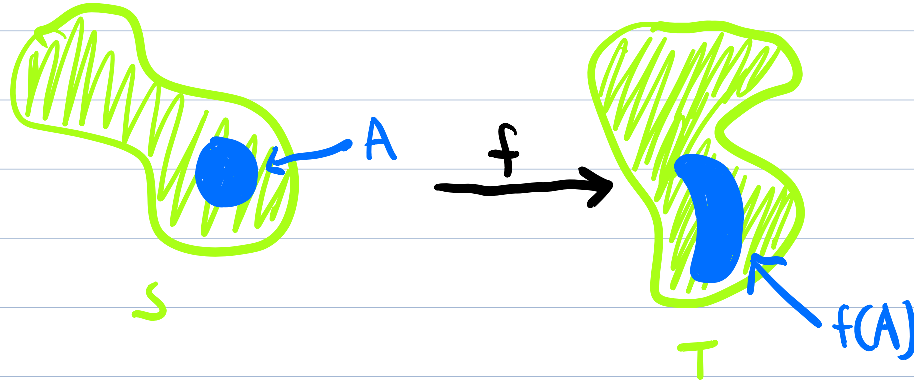
$$(f\circ f^{-1})_* = \underset{Id^{\pi(T)}}{\underset{\|}{Id^T_*}} \Rightarrow f_*\circ f^{-1}_* = Id^{\pi(T)}.$$

Thus, $f_*: \pi(S) \to \pi(T)$ is invertible, with inverse f^{-1}_*. Therefore f_* is a bijection. $\blacksquare$

Application: We prove that as unions of curves w/ no thickness, X and Y are not homeomorphic.

[The argument will be skipped in class]

Fact: If $f: S \to T$ is a homeomorphism and $A \subset S$, then A and $f(A)$ are homeomorphic, where $f(A) = \{ y \in T \mid y = f(x) \text{ for some } x \in A \}$.



<u>proof of fact</u>: Let $j: A \to S$ be the inclusion. $\mathrm{im}(f\circ j) = f(A)$. Since f is a bijection, so $\widetilde{f\circ j}: A \to f(A)$. It follows from the facts about continuity stated in an earlier lecture that $\widetilde{f\circ j}$ is continuous. Moreover, if $j': f(A) \to T$ is the inclusion, $(\widetilde{f\circ j})^{-1} = \widetilde{f^{-1}\circ j'}$, and this is continuous by the same reasoning.

<u>Proof that X and Y are not homeomorphic</u>:
Let $X' \subset X$ be obtained by removing the center point p. $|\pi(X')| = 4$. Note that there no way to remove a single point from Y to get $Y' \subset Y$ with $|\pi(Y)| = 4$

If we have a homeomorphism $f: X \to Y$, then $f(X')$ is obtained from Y by removing $f(p)$, and $|\pi(f(X'))| = |\pi(X')| = 4$ by the prop., which is impossible. Thus, no homeomorphism

$f: X \to Y$ can exist.

Topology Beyond Subsets of Euclidean Space

So far in this course, we've only considered continuity of functions $f: S \to T$ where S and T subsets of Euclidean spaces.

we sometimes use the word "subspace"

Hence, all the topological concepts we've introduced so far, e.g.,

- homeomorphism
- isotopy
- path components

have been defined in class only for Euclidean subspaces.

However, these ideas make sense in much more generality, and that extra generality can be extremely useful.

In fact, there are two levels to this extra generality. We discuss first level now.

Recall our definition of a continuous function between Euclidean subspaces:

Formal Definition of Continuity

We say $f: S \to T$ is <u>continuous</u> at $x \in S$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in S$ and $d(x,y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

We say f is <u>continuous</u> if it is continuous at all $x \in S$.

from Lec 3

Important observation: The only way we are using the fact that S and T are Euclidean subspaces is through their distance functions.

$\Rightarrow$ Continuity should make sense for any functions between sets endowed with some reasonable definition of a distance.

There are many extremely important examples, beyond the Euclidean subspaces we've already seen.

To explain this formally, we introduce metric space's.

A metric space is a set S, together with a function $d: S \times S \to [0,\infty)$ satisfying:

1) $d(x,y)=0$ if and only if $x=y$.
2) $d(x,y)=d(y,x)$ [symmetry]
3) $d(x,z) \le d(x,y)+d(y,z)$ $\forall\ x,y,z \in S$ [triangle inequality].

We denote the metric space as (S,d). We call d a metric.

Example: The familar example: $S=\mathbb{R}^n$, $d_2: \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$,

$$d_2(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}$$

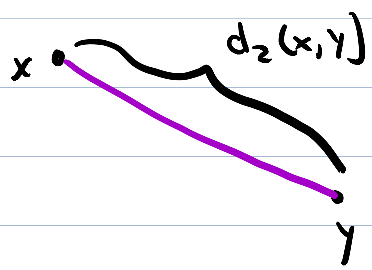
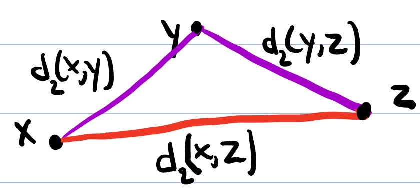


Illustration of the triangle inequality (case that x,y,z don't all lie on the same line)



$d_2(x,z) \le d_2(x,y) + d_2(y,z)$ because the length of any side of a triangle is less than the sum of the lengths of the other two sides. Hence the name "triangle inequality."