AMAT 583 Lec 14 10/10/19

Today: Open sets and continuity
Homotopy Equivalence

a definition of topological equivalence weaker than homeomorphism, but still very useful.

Review: The idea of continuity of a function $f: S \to T$ makes sense whenever S and T are metric spaces.
(S and T needn't be assumed to be subsets of $\mathbb{R}^n$).

Hence, basic topological concepts like homeomorphism extend to metric spaces.

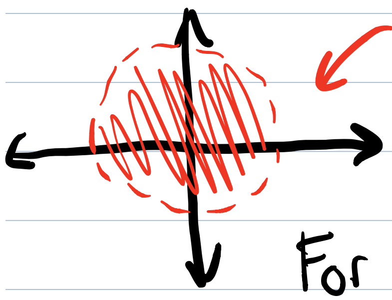
An alternate discription of continuity

Open Sets

Let M be a metric space. For $x \in M$ and $r > 0$, the open ball in M of radius r, centered at x, is the set

$$B(x,r) = \{y \in M \mid d_M(x,r) < r\}.$$

Example: For $M = \mathbb{R}^2$ with the Euclidean distance. $B(\vec{0}, 1)$ looks like this



disc of radius 1 centered at the origin, with the boundary not included.

For $M = \mathbb{R}$, the open ball of radius r centered at x is just the interval $(x-r, x+r)$.

A subset of M is called <u>open</u> if it is a union of (possibly infinitely many) open balls.

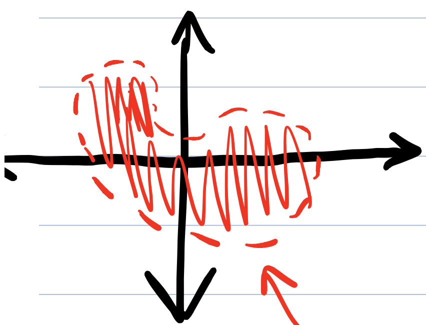
The <u>empty set</u> is always considered open.

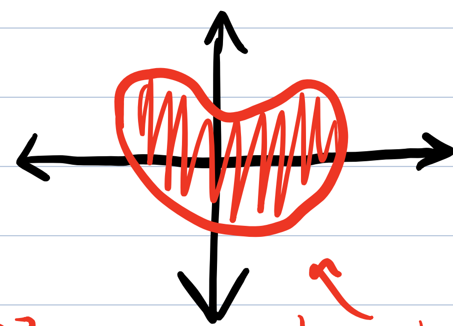
M itself is open: $M = \bigcup_{x \in M} B(x, 1)$

<u>Fact</u>: A region in $\mathbb{R}^n$ is open if it contains none of its boundary points.

this is an informal statement because I haven't defined "boundary points." It can be made formal, but I will not go into the details.

Illustration: Dashed line = boundary not included
Solid line = boundary included

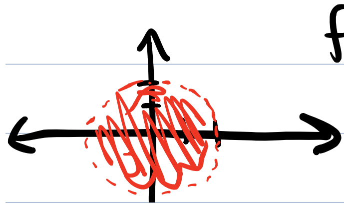




Fundamental Fact: Whether a function of of metric spaces $f: M \to N$ is continuous depends only on the open sets of M, N and not otherwise on the metric! (this is made precise by the proposition below)

Notation: For $f: S \to T$ any function and $U \subset T$, $f^{-1}(U) = \{x \in S \mid f(x) \in U\}$.

Example: Let $f: \mathbb{R}^2 \to [0, \infty)$ be given by $f(x) = d_2(x, 0)$.



$f^{-1}([0,2))$ = the open ball of radius 2 centered at 0.

↓

Proposition: A function $f: M \to N$ of metric spaces is continuous if and only if $f^{-1}(U)$ is open for every open subset of N.

Proof: Exercise.

Philosophical implications:

- In topology, we study geometric objects via the continuous functions between them.
  (The continuous functions are what matter in topology)
- Thus, in view of the proposition, the specific choice of metric on a metric space matters topologically only insofar as this determines the open subsets of the metric space.

This motivates the following definition:

Def:
Two metrics d_1 and d_2 on a set S are called topologically equivalent if

(S, d_1) and (S, d_2) have the same open sets.

Interpretation: Topologically equivalent metrics look the same through the lens of topology.

Note: Examples of topologically equivalent metrics are common.

Fact: If there are positive constants $0 < \alpha, \beta$ such that $\forall x, y \in S$, $\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_2(x,y)$, then d_1 and d_2 are topologically equivalent.

Example: Recall that for $p \in [1, \infty)$, we defined the metric

$$d_p \text{ on } \mathbb{R}^n \text{ by } d_p(x,y) = \sqrt[p]{\sum_{i=1}^{n} |x_i - y_i|^p}.$$

Well known fact: For all $p, q \in [1, \infty)$, d_p and d_q are topologically equivalent.

Example: The intristic and extrinsic metric on S^1 are topologically equivalent.

Metrics on Cartesian Products

Let M and N be metric spaces with metrics d_M d_N.

How do I define a metric d on $M \times N$?

Note: This question is important, because to extend the definition of homotopy to metric spaces, we need to talk about a continuous function $h: M \times I \to N$ where M, N are metric spaces. But then we need a metric structure on $M \times I$

There are multiple options, e.g.

- $d((m_1, n_1), (m_2, n_2)) = d_M(m_1, m_2) + d_N(n_1, n_2)$
  $d((m_1, n_1), (m_2, n_2)) = \max(d_M(m_1, m_2) + d_N(n_1, n_2))$

But it turns out that these are topologically equivalent!

Topology without Metrics (Abstract Formulation of Topology)

Idea: If all that matters in topology is the open sets of a metric space, then perhaps it would be simpler to give the basic definitions of topology without mentioning metrics at all.

Key properties of open subsets of a metric space M

1) Union of open sets is open
2) Intersection of finitely many open sets is open
3) M is open
4) $\emptyset$ is open.

Lets use these facts as inspiration to make a definition

Definition: A topological space is a pair (T, O), where:

- T is a set
- O is a collection of subsets of T, called the open sets, satisfying properties 1)-4) above

Note that there is no mention of a metric here.

Definition: A function $f: X \to Y$ between topological spaces is called **continuous** if $f^{-1}(U)$ is open whenever U is open.

Example: For any metric space (S, d), call the collection of open sets $\mathcal{O}$. Then $(S, \mathcal{O})$ is a topological space.

Note: Most examples of topological spaces arise via metrics, as in the above example. But not all do.

From now on, we use the language of topological spaces, but for concreteness, you can think of metric spaces, or subsets of Euclidean spaces.

Homotopy equivalence

Motivation: Two spaces may not be homeomorphic, but may be topologically similar in a looser sense. We would like to quantify this.