AMAT 583 Lec 15 10/17/19

<u>Today</u>: Topological Spaces,
Homotopy Equivalence,
Clustering

Overview of last class

- We observed that the ϵ-δ def. of continuity extends to functions between metric spaces.

- We noted that a function $f: M \to N$ of metric spaces iff $f^{-1}(U)$ is open for each open set $U \subset N$.

A set $U \subset M$ is <u>open</u> if it is a union of open balls. (Here M is any metric space).

$\Longrightarrow$ That continuity depends only on the open sets of M and N, not otherwise on the metrics.

$\Longrightarrow$ Topology depends only on the open sets!

- **Def**: If d_1 and d_2 are metrics on the same set S and d_1 and d_2 have the same collection of open sets, then d_1 and d_2 are called topologically equivalent.

Topologically equivalent metrics are <u>common</u>.

- Can develop topology without talking about metrics at all, and just tracking open sets!

Definition: A <u>topological</u> <u>space</u> is a set T, together with a collection $\mathcal{O}$ of subsets of T

1) $\mathcal{O}$ is closed under arbitrary unions
2) $\mathcal{O}$ is closed under finite intersections
3) $T \in \mathcal{O}$
4) $\phi \in \mathcal{O}$.

$\mathcal{O}$ is called a <u>topology</u> on T or a <u>topological</u> <u>structure</u>.

Definition: A function $f: T \to T'$ of topological spaces (with respective collections of open sets $\mathcal{O}, \mathcal{O}'$) is <u>open</u> if $f^{-1}(U) \subset \mathcal{O}$ for all $U \subset \mathcal{O}'$!

That is, if $f^{-1}(U)$ is open for each open set $U \subset T'$.

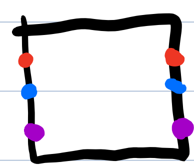
Example: For any metric space (S, d), let $\mathcal{O}$ denote the collection of open sets. $(S, \mathcal{O})$ is a topological space.

Note: Most but not all topological spaces one encounters arise from a metric.

What is this more abstract perspective good for?

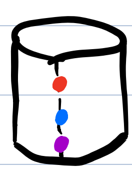
One answer: gluing.

Consider the square $I \times I$



Thinking of this as a piece of rubber, suppose we glue the left edge to the right edge, i.e., gluing $(0, y)$ to $(1, y)$ for all $y \in I$

We get a cylinder:



How do we model such gluing mathematically?
We can put a metric on the glued object, but this is awkward.
It's much cleaner to work directly with open sets.

Quotient Topology

For T a topological space, and $\sim$ an equivalence relation on T, define a topology on $T/\sim$ by taking $U \subset T/\sim$ to be open iff $\pi^{-1}(U)$ is open, where

$$\pi : T \to T/\sim \text{ is given by } \pi(x) = [x].$$

$T/\sim$, together with this topology, is called a quotient space.

This definition may be a bit mysterious, but it has an intuitive interpretation.

This topology is the one on $T/\sim$ obtained from T by gluing together as little as possible.

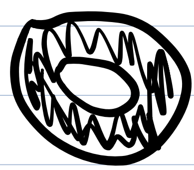
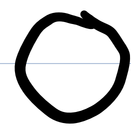
Example: Define an equivalence relation on $I \times I = I^2$ by $(a,b) \sim (c,d)$ iff 1) $b = d$ AND
2) $a = c$ OR
$a = 0, c = 1$ OR
$a = 1, c = 0$.

Then the quotient space $T/\sim$ is homeomorphic to $S^1 \times I$.

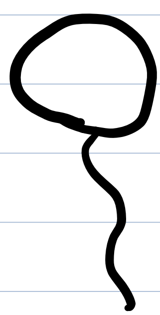
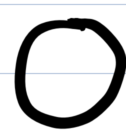
Homotopy equivalence

Motivation: Two spaces may not be homeomorphic, but may be topologically similar in a looser sense. We would like to quantify this.

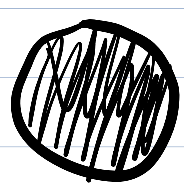
Examples

 vs. 

Annulus — Circle

 vs. 

Circle w/ "rat tail" — Circle

 vs 

Disk — Point

Each pair of spaces is not homeomorphic, but is homotopy equivalent.

Loosely speaking, homotopy equivalent spaces has same number of holes of different types.

Product topology (technical detail)

$X = (S^X, O^X)$ $Y = (S^Y, O^Y)$

For topological spaces and , the product space $X \times Y$ is the topological space w/ underlying set $S^X \times S^Y$ and $U \subset S^X \times S^Y$ open iff

U is a union of sets of the form

$U \times V$, where $U \subset O^X$ and $V \subset O^Y$.

Note: We talked last time about different ways to put a metric on a Cartesian product of metric spaces.

Each way we discussed yields the product topology. That's the motivation for this definition.

Homotopy Equivalence

Deformation retract