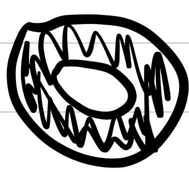
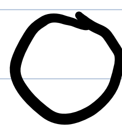
AMAT 583 Lec 16 10/24/19

Today: homotopy equivalence
clustering

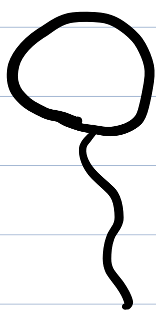
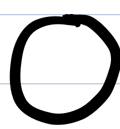
Homotopy equivalence (review from last time)

Motivation: Two spaces may not be homeomorphic, but may be topologically similar in a looser sense. We would like to quantify this.

Examples

 vs. 

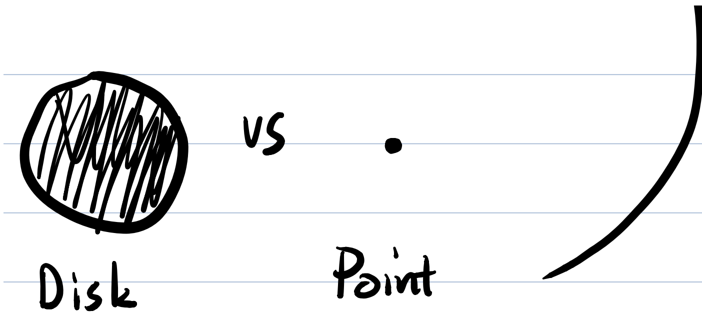
Annulus — Circle

 vs. 

Circle w/ "rat tail" — Circle

Each pair of spaces is not homeomorphic, but is homotopy equivalent.

Loosely speaking, homotopy equivalent spaces has same number of



holes of different types.

<u>Recall</u>: For $f, g : S \to T$ continuous maps, a homotopy from f to g is a continuous map
$h : S \times I \to T$
such that $h_0 = f$, $h_1 = g$, where $h_t : S \to T$ is given by $h_t(x) = h(x,t)$.

[here, S and T are topological spaces, but you are welcome to think of them as metric spaces or subsets of $\mathbb{R}^n$, if you prefer.]

If f is homotopic to g, we write $f \sim g$.

<u>Fact</u>: $\sim$ is an equivalence relation on $C(S,T)$, the set of continuous functions from S to T.

In particular, a continuous map $f : S \to T$ is always homotopic to itself: Take $h : S \times I \to T$ to be given by $h(x,t) = f(x)$ $\forall\, t \in I$. This is a homotopy from f to f.

Thus, $f = g \Rightarrow f \sim g$.

Def: A <u>homotopy equivalence</u> is a continuous map of topological spaces $f: S \to T$ s.t.

$\exists$ continuous $g: T \to S$ with

$g \circ f \sim Id_S \quad f \circ g \sim Id_T$.

g is called the homotopy inverse of f.

Proposition: Any homeomorphism is a homotopy equivalence.

Proof: If $f: S \to T$ is a homeorphism then f has a continuous inverse $g: T \to S$, so

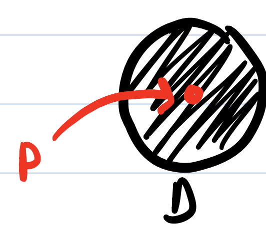
$$g \circ f = Id_S, \quad f \circ g = Id_T$$

$$\Rightarrow g \circ f \sim Id_S, \quad f \circ g \sim Id_T. \quad \blacksquare$$

Example: Consider

the disk $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

$P = \{(0,0)\}$.



Let $f: D \to P$ denote the constant map,
i.e. $f(x) = (0,0)\ \forall x \in D$.

Let $g: P \to D$ be the inclusion.

Then $f \circ g = Id_P$

$g \circ f : D \to D$ is the constant map to $(0,0)$.

We define a homotopy h from Id_D to $g \circ f$.

$h: D \times I \to D$ by

$h(\vec{x}, t) = (1-t)\vec{x}$.

Clearly $h_0 = Id_D$, and $h_1 =$ the constant map to $(0,0)$.

Thus $Id_D \sim g \circ f$.

$\Rightarrow$ f and g are inverse homotopy equivalences.

__Definition__: If a topological space X is homotopy equivalent to a point, we say X is __contractible__.

Thus, $D^\bullet$ is contractible.

Intuitively, a space is contractible iff it has no holes.

Exercise Regarding the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 unions of curves with no thickness, which are contractible?

A: 1, 2, 3, 4, 7

Proposition: If $f: S \to T$ is a homotopy equivalence, and $g: T \to U$ is a homotopy equivalence, then $g \circ f$ is a homotopy equivalence.

Pf: Exercise, or see Hatcher, Ch. 0.

Deformation Retracts

So far, homotopy equivalence is a mysterious relation. I will give a more intuitive interpretation.

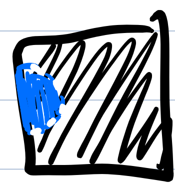
First, we introduce a special kind of homotopy called a deformation retraction.

Let X be a topological space and $A \subset X$ a subset.

<u>Technical</u> <u>remark</u>: A is also a topological space, if we take the open sets of A to be the intersection of open sets of X with A.

<u>Example</u>:

open in A but not in X.



$X = \mathbb{R}^2$,
$A = I \times I$

<u>Def</u>: A continuous map $h: X \times I \to X$ is a <u>deformation retraction</u> of X onto A if $h_0 = Id_X$, $\mathrm{im}(h_1) \subset A$, and $h(y,t) = y$ $\forall (y,t) \in A \times I$.

<u>Note</u>: h is a homotopy.

<u>Example</u>: We already saw a deformation retraction above:

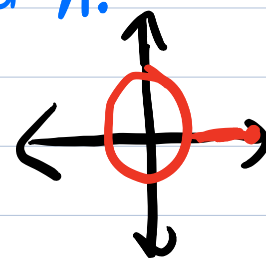
$$h: D \times I \to D, \quad h(\vec{x}, t) = (1-t)\vec{x}$$

Take $X = D$
$A = P = \{0,0\}$.

Intuitively, h shrinks D down to the point P.
That is, $\mathrm{im}(h_0) = D$, $\mathrm{im}(h_1) = P$.

In general, a deformation retraction from X onto A shrinks X down onto A, without moving any point of A.

<u>Example</u> $X = S^1 \cup [1,2] \times \{0\}$
$A = S^1$



$h : X \times I \to A$, $h(x,t) = \begin{cases} x & \text{for } x \in S^1 \\ ((x-1)(1-t)+1, 0) & \text{otherwise.} \end{cases}$

Then h is a deformation retract of X onto A.

This shrinks the rat tail down onto $(1,0)$.
(continuity is not hard to check).

<u>Fact</u>: If $\exists$ a deformation retract

$h : X \times I \to A$ of X onto A, then

for $j : A \hookrightarrow X$ the inclusion,

j and $\tilde{h}_1 : X \to A = \mathrm{im}(h_1)$ are inverse homotopy equivalences:

$h_1 \circ j = \mathrm{Id}_A$ and h is a homotopy from

Id_X to $h_1 = j \circ \tilde{h}_1$, so $j \circ h_1 \sim Id_X$.