AMAT 583, Lec 17, 10/29/19

__Today:__ Finish with homotopy equivalence
Examples of gluing
Data analysis (clustering) (Next time)

__Recall:__ A continuous map
$f: S \to T$ is called a __homotopy equivalence__
if there exists a continuous map
$g: T \to S$ such that

$$g \circ f \sim Id_S \qquad f \circ g \sim Id_T$$

denotes homotopy

In this case, we say S and T are homotopy equivalent, and write $S \simeq T$.

__Idea from the end of last class:__
S and T are homotopy equivalent iff $\exists$ a third space U that can be "shrunk down" to both S and T.

__Illustration:__ The disc $D \subset \mathbb{R}^2$ shrinks down to a point

Def: A continuous map $h: X \times I \to X$ is a deformation retraction of X onto $A \subset X$ if

We think of h as specifying how to shrink X onto A continuously in time

1. $h_0 = Id_X$, ⟵ At time 0, x has not been shrunk at all
2. $im(h_1) = A$ ⟵ h shrinks X onto A by time 1.
3. and $h(y,t) = y$ $\forall (y,t) \in A \times I$ ⟵ we don't move any point of A.

Note: h is a homotopy between Id_X and a map $h_1: X \to X$ with $im(h_1) = A$.

Example: We already saw a deformation retraction last time

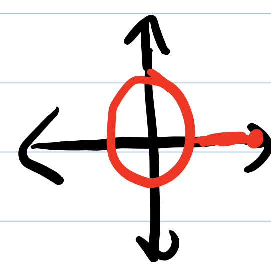
$$h: D \times I \to D, \quad h(\vec{x}, t) = (1-t)\vec{x}$$

Take $X = D$
$A = P = \{0,0\}$.

$h_0(\vec{x}) = (1-0)\vec{x} = \vec{x} \Rightarrow h_0 = Id_X$ 1.
$h_1(\vec{x}) = (1-1)\vec{x} = 0 \quad \Rightarrow im(h_1) = P = \{\vec{0}\}$. 2.
$h_t(0) = (1-t)0 = 0 \quad \forall t \in I \Rightarrow$ 3.

Intuitively, h shrinks D down to the point P.
That is, $im(h_0) = D$, $im(h_1) = P$.

Example $X = S^1 \cup [1,2] \times \{0\}$
$A = S^1$



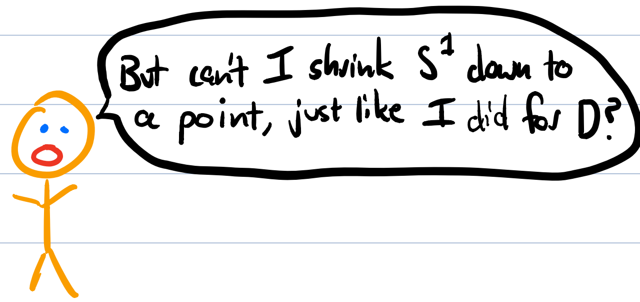
$$h : X \times I \to A, \quad h(x,t) = \begin{cases} x & \text{for } x \in S^1 \\ ((x-1)(1-t)+1,\, 0) & \text{otherwise.} \end{cases}$$

Then h is a deformation retract of X onto A.

This shrinks the rat tail down onto $(1,0)$. (continuity is not hard to check).

<u>Example</u>: Are S^1 and a point P homotopy equivalent?

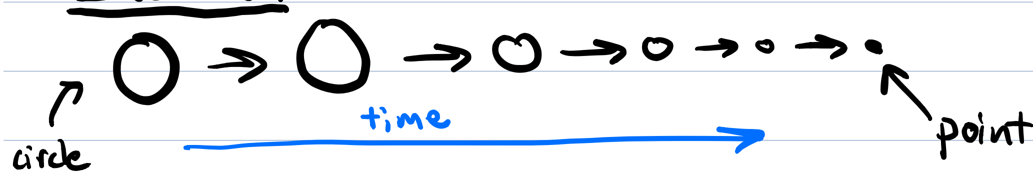
No!



thoughtful student

No! As the definition of deformation retraction makes clear, the shrinking has to happen entirely inside of $X = S^1$, i.e. we would need $\mathrm{im}(h) \subset S^1$. So the devious idea of just making the circle smaller and smaller until it becomes a point fails.

<u>Illustration</u>:



Note: It can be shown rigorously that there's no deformation retract from S^1 onto a point. This is usually done by considering holes, e.g. via homology.

<u>Fact</u>: If $\exists$ a deformation retract

$h: X \times I \to X$ of X onto A, then

for $j: A \hookrightarrow X$ the inclusion,

j and $\tilde{h}_1 : X \to A = \mathrm{im}(h_1)$ are inverse homotopy equivalences.

$\Longrightarrow$ X and A are homotopy equivalent.

<u>Proof</u>: (Not covered in class)

$\tilde{h}_1 \circ j = Id_A$ by property 3, so $\tilde{h}_1 \circ j \sim Id_A$.

Conversely, for <u>any</u> function $f: S \to T$ and $j: \mathrm{im}(f) \hookrightarrow T$ the inclusion, we have $f = j \circ \tilde{f}$. Thus, since $\mathrm{im}(h_1) = j$, we have $j \circ \tilde{h}_1 = h_1$.
h is a homotopy from Id_X to $h_1 = j \circ \tilde{h}_1$, so $j \circ h_1 \sim Id_X$. ∎

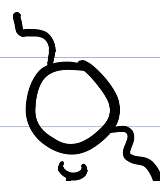
<u>Example</u>: D and P are homotopy equivalent, since D deformation retracts onto P. We also saw this earlier.

Fact: Topological spaces X and Y are homotopy equivalent iff $\exists$ a third space Z which deformation retracts onto both X and Y

Part of a proof:

If Z def. retracts onto X and Y, then $Z \simeq X$ and $Z \simeq Y$ so $X \simeq Y$.

The converse direction is more difficult. See Hatcher Ch. 0. $\blacksquare$

Example:  and  both deformation

X $\quad$ Y

retract down to a circle, so they are homotopy equivalent. This implies both are deformation retracts of some larger space Z.
For example we may take $Z =$ the annulus 

Illustration of a deformation retraction of Z onto X.

Can draw a similar picture for Y

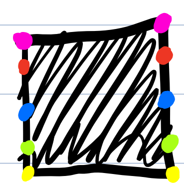
Gluing revisited

In an earlier lecture, I said I show some interesting examples of gluing constructions but then I forgot!

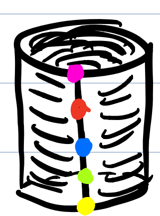
Before moving on to clustering, I want to give a few more famous examples of gluing constructions.

These can be specified formally by the quotient space construction we saw earlier, but I will be informal.

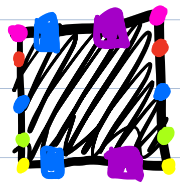
1. Recall the example from earlier



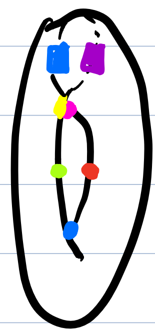
If we start with the square $I \times I$ and glue $(0,y)$ to $(1,y)$ $\forall\, y \in I$, we get the cylinder



2. What if we also glue the top edge to the bottom edge, i.e. glue

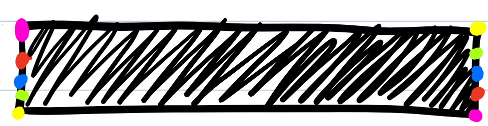


$(x,0)$ to $(x,1)$



We get a torus, i.e. surface of a donut.

3. What if in the first example, we instead glue $(0,y)$ to $(1,1-y)$



We get the Mobius band

This is a surface with one side!