Today: Sets + Functions, continued

Cartesian Products: ← We just started this last time.

Definition: For sets S and T, the Cartesian product of S and T, denoted $S \times T$, is the set of all ordered pairs (s,t) with $s \in S$ and $t \in T$.

In symbols, we write this as:

$$S \times T = \{(x,y) \mid x \in S,\ y \in T\}.$$

also in last lec. notes

Note: Here we are using parentheses to denote an ordered pair (s,t). But just before, we used parentheses to denote an open interval.

These are the notational conventions that are typically used. It is a bit unfortunate that the same notation is used for two different things. In practice, though, this rarely causes confusion, as it's usually clear from context what is meant.

Example: For $S = \{1,2\}$ and $T = \{a,b\}$,

$S \times T = \{(1,a),(1,b),(2,a),(2,b)\}.$

<u>Example</u>: By definition, $\mathbb{R} \times \mathbb{R}$ is the set of ordered pairs of real numbers.

We denote $\mathbb{R} \times \mathbb{R}$ as $\mathbb{R}^2$.

More generally, given sets $S_1, S_2, \ldots, S_n$,

The Cartesian product

$S_1 \times S_2 \times \cdots \times S_n$ is the set of

ordered lists $(x_1, x_2, \ldots, x_n)$ where $x_i \in S_i$ for each i.

In symbols,

$$S_1 \times S_2 \times \cdots \times S_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in S_i \ \forall i\}.$$

this symbol means "for all."

Example: For T any set, we denote

$$\underbrace{T \times T \times \cdots \times T}_{n \text{ copies of } T} \text{ by } T^n.$$

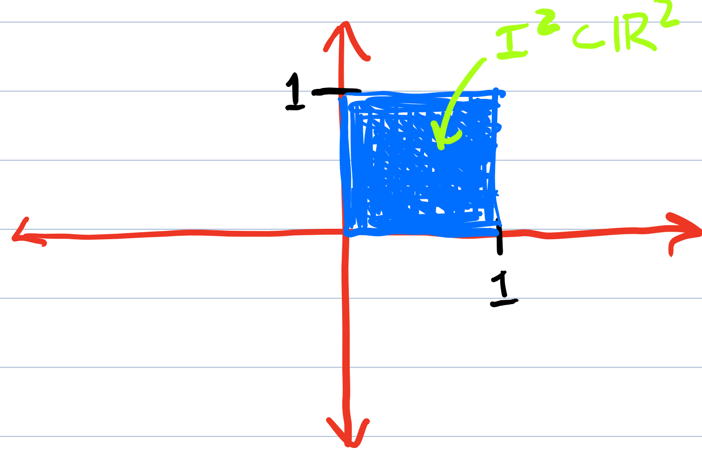
In particular, this gives a definition of $\mathbb{R}^n$ as a set:

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies of } \mathbb{R}} = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R} \ \forall i\}.$$

Example

I^n is called the n-dimensional unit cube.

Illustration for $n = 2$



Exercise: What are the elements of $\{0,1\}^2 \subset \mathbb{R}^2$.

What is the geometric relationship between $\{0,1\}^2 \subset \mathbb{R}^2$ and I^2?

Union of Sets

The union of two sets S and T, denoted $S \cup T$, is the set consisting of all elements in either S or T.

Example: If $S = \{a,b\}$ and $T = \{b,c\}$, then

$$S \cup T = \{a,b,c\}.$$

Exercise: If $S = \{a,b\}$, what is $S \cup S$?
what is $S \cup \emptyset$?

Intersection of Sets

The intersection of sets S and T, denoted $S \cap T$, is the set consisting of all elements in both S and T.

Example: For S and T as in the example above,
$S \cap T = \{b\}$.

Exercise: If $S = \{a,b\}$, what is $S \cap \emptyset$?

Complements: Given sets $S \subset T$, the complement of S in T, denoted $S^c \subset T$ is the set of all elements in T not contained in S.

That is $S^c = \{x \in T \mid x \notin S\}$.

Example: If $S = \{1,3,5\}$ and $T = \{1,2,3,4,5\}$ $S^c \subset T$ is $\{2,4\}$.

Example: $[0,1]^c \subset \mathbb{R}$ is

$$(-\infty, 0) \cup (1, \infty) = \{x \in \mathbb{R} \mid x < 0 \text{ or } x > 1\}.$$

Exercise: What is $(2,\infty)^c \subset \mathbb{R}$?

Next topics:
- Functions
- Continuous functions
- Homeomorphisms

Definition: Given sets S and T, a function f from S to T is a rule which assigns each $s \in S$ exactly one element in T.
- This element is denoted $f(t)$.

We call
S the domain of f.
T the codomain of f.

We write the function as $f: S \to T$.

Example: Let $S = \{1, 2\}$, $T = \{a, b\}$.
e can define a function $f: S \to T$ by $f(1) = a$, $f(2) = b$
$g: S \to T$ by $f(1) = a$, $f(2) = a$.

Example: We often specify a function by a formula, e.g.
$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. OR
$f: \mathbb{R} \to \mathbb{R}$, $f(x) = 5e^{-x^2}$

For $f: S \to T$ and $g: T \to U$, the composite $g \circ f: S \to U$ is the function given by $g \circ f(x) = g(f(x))$.

Ex: S, T, f as above, $U = \{x, y, z\}$, $h: T \to U$, $h(a) = x$, $h(b) = z$ $\quad h \circ f(1) = x$, $h \circ f(2) = z$

Image of a function (also called the range)

Definition: For a function $f: S \to T$ we define $\mathrm{im}(f)$ to be the subset of T given by

$$\mathrm{im}(f) = \{t \in T \mid t = f(s) \text{ for some } s \in S\}.$$

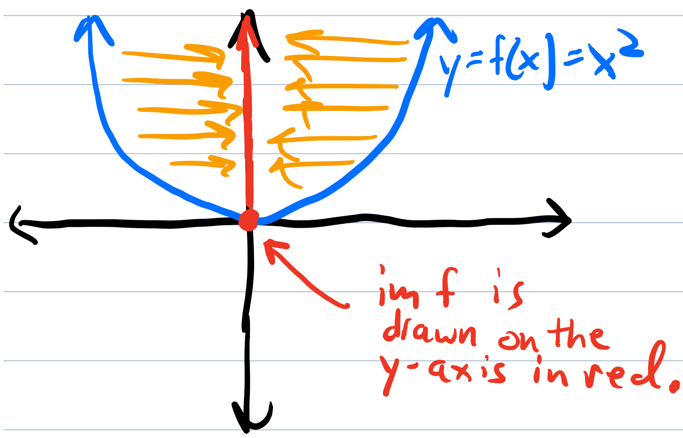
Intuitively, $\mathrm{im}(f)$ is the subset of T consisting of elements "hit by" f.

Example: For S, T, f, and g as in the previous example,

$$\mathrm{im}(f) = \{a, b\} = T, \quad \mathrm{im}(g) = \{a\}.$$

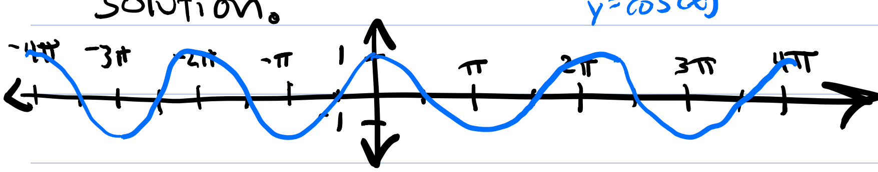
Example:

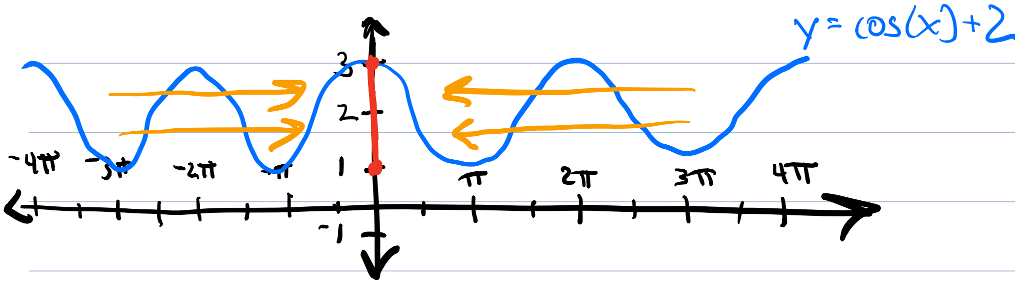
for $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$,
$\operatorname{im} f = [0, \infty)$



Exercise: For $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \cos(x) + 2$, what is $\operatorname{im} f$?

solution.





$\operatorname{im}(\cos) = [-1, 1]$, so $\operatorname{im}(f) = [1, 3]$

Exercise:

Let $f: \mathbb{R} \to \mathbb{R}^2$ be given by
$f(x) = (\cos x, \sin x)$.

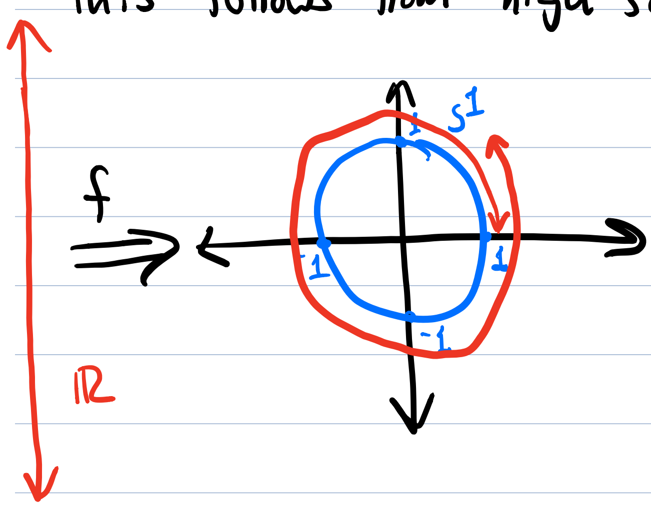
point y on the unit circle such that $\vec{0y}$ makes angle x (in radians with the positive x-axis).

What is $\operatorname{im} f$?

Solution: $\operatorname{im} f = S^1$, where S^1 denotes the unit circle, i.e.,

$$S^1 = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1\}.$$

This follows from high school trig.



f "wraps" $\mathbb{R}$ around S^1 an infinite number of times

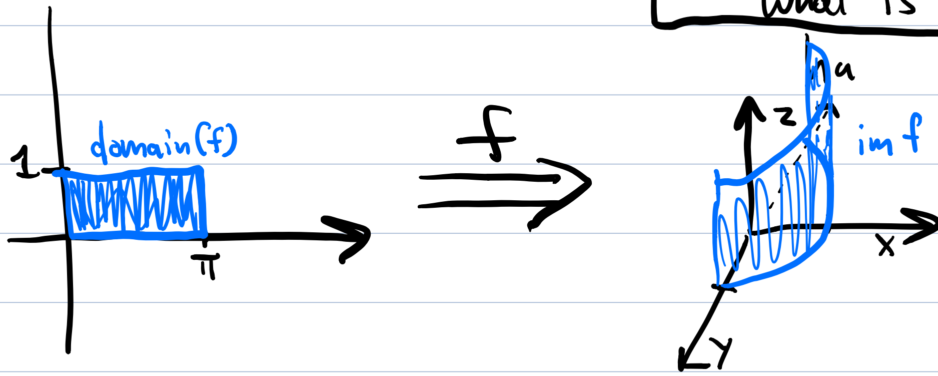
Example: Let $f: [0,\pi] \times I \to \mathbb{R}^3$ be given by
by $f(x,y) = (\cos x, \sin x, y)$

$\operatorname{im}(f)$ is a half-cylinder.

[lecture ended here.]

first consider
$g: [0,\pi] \to \mathbb{R}^2$, given by
$g(x,y) = (\cos x, \cos y)$.

What is im g?



Useful Notation: For $f: S \to T$ a function and $U \subset S$, $f(U) = \{ y \in T \mid y = f(x) \text{ for some } x \in U \}$.

Note: $f(S) = \text{im}(S)$. In general, $f(U) \subset \text{im}(S) \subset T$.

Injective, Surjective, and Bijective Functions

We say a function $f: S \to T$ is

injective (or 1-1) if $f(s) = f(t)$ only when $s = t$.

surjective (onto) if $\text{im}(f) = T$.

bijective (a bijection) if f is both injective and surjective.

Example: $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is neither injective nor surjective.

Example $f: \mathbb{R} \to S^1$ given by

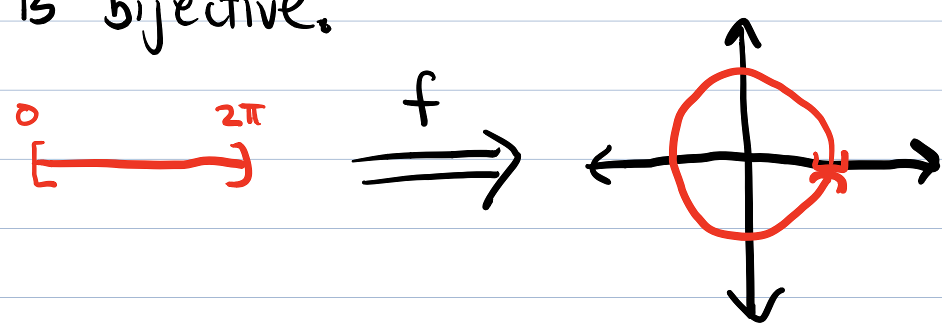
$f(x) = (\cos x, \sin x)$ is surjective but not injective.

e.g. $f(0) = f(2\pi) = (1,0)$.

Example $f: [0, 2\pi) \to S^1$ given by

$f(x) = (\cos x, \sin x)$

is bijective.



Bijections and Inverses

For S any set, the *identity function* on S, is the function

$Id_S : S \to S$ given by $Id_S(x) = x \ \forall x \in S$