AMAT 583 Lecture 4

Functions, Continuous functions

Recall: for $f: S \to T$, $\mathrm{im}(f) = \{y \in T \mid y = f(x) \text{ for some } x \in S\}$.

We can also talk about the image of subsets of a function:

For $U \subset S$ and $f: S \to T$,

$$f(U) = \{y \in T \mid y = f(x) \text{ for some } x \in U\}.$$

Note: $f(S) = \mathrm{im}(f)$.

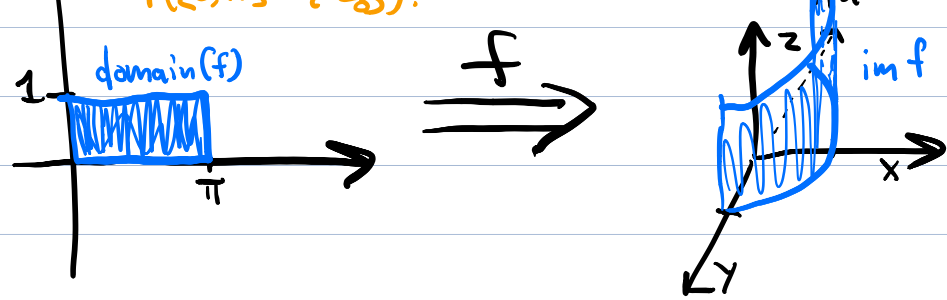
Example: Let $f: [0,\pi] \times I \to \mathbb{R}^3$ be given by
by $f(x,y) = (\cos x, \sin x, y)$

Question: What is $\mathrm{im}(f)$?

Ans: $\mathrm{im}(f)$ is a half-cylinder.

First question: Consider $g: [0,\pi] \to \mathbb{R}^2$, given by $g(x,y) = (\cos x, \cos y)$. What is $\mathrm{im}\, g$?

Second question: What is $f([0,\pi] \times \{z\})$?



Injective, Surjective, and Bijective Functions

We say a function $f: S \to T$ is

injective (or 1-1) if $f(s) = f(t)$ only when $s = t$.

surjective (onto) if $\text{im}(f) = T$.

bijective (a bijection) if f is both injective and surjective.

Example: $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$
is neither injective nor surjective.

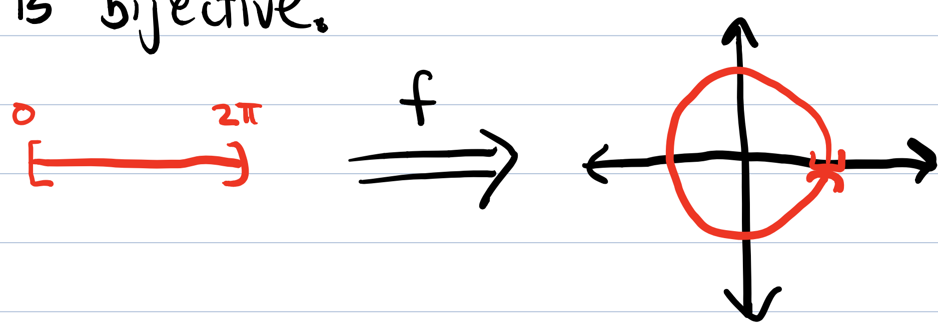
Example $f: \mathbb{R} \to S^1$ given by
$f(x) = (\cos x, \sin x)$ is surjective but not injective.

e.g. $f(0) = f(2\pi) = (1, 0)$.

Example $f: [0, 2\pi) \to S^1$ given by

$f(x) = (\cos x, \sin x)$

is bijective.



Bijections and Inverses

For S any set, the <u>identity function</u> on S is the function

$Id_S : S \to S$ given by $Id_S(x) = x \;\; \forall x \in S$.

Fuctions $f: S \to T$ and $g: T \to S$ are said to be <u>inverses</u> if

$\underbrace{g \circ f}_{\text{function composition}} = Id_S$ and $f \circ g = Id_T$.

We call g the inverse of f, and write g as f^{-1}.

Fact: A function $f: S \to T$ has an inverse $g: T \to S$ if and only if f is a bijection.

($g(y)$ is the unique element $x \in S$ with $f(x) = y$.)

Example Let $f: [0, 2\pi) \to S^1$ be bijection of the previous example.

We define the inverse $g: S^1 \to [0, 2\pi)$ to be the function which maps $y \in S^1$ to the angle θ $\overrightarrow{Oy}$ makes with the positive x-axis (in radians).

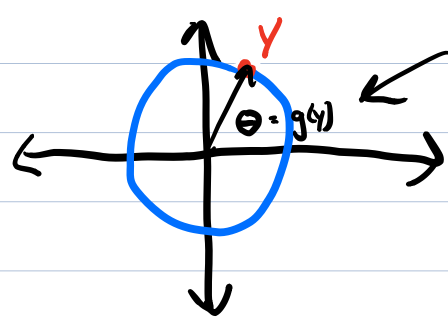
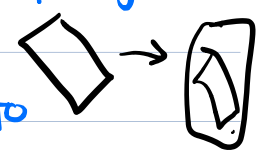


Illustration of g.

Continuous functions (An essential notion in topology.)

Geometrically, we can think of a function $f: S \to T$ as "putting S inside T." For example, S could be a piece of paper, and T could be my book-bag. Then f specifies how I put the paper in my bag.



Of course, unless f is injective, f is allowed to have two different points of the paper go to the same point in the bag, or to make the paper pass through itself.

In general f can put the paper in the bag in a way that <u>shreds the paper to bits.</u>

Informally, a function $f: S \to T$ that "puts S into T without tearing S" is a continuous function.

To talk about the continuity of a function $f: S \to T$, we need some way of

- measuring distances between points in S
- measuring distances between points in T.

(S, T need some additional structure beyond just being sets.)

(Actually, we need a bit less than this to talk about continuity, but that is a point that we will return to later.)

To start, let's consider the continuity of functions $f: S \to T$ where $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$
$y = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

let $d(x,y)$ denote the Euclidean distance between x and y, i.e.,

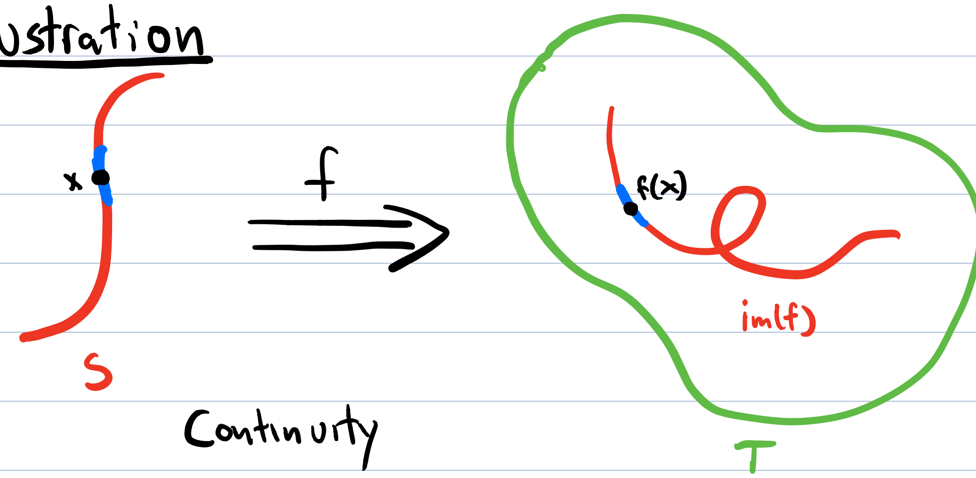
$$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2} = \|x-y\|.$$

Note: This defines a function $d: \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$

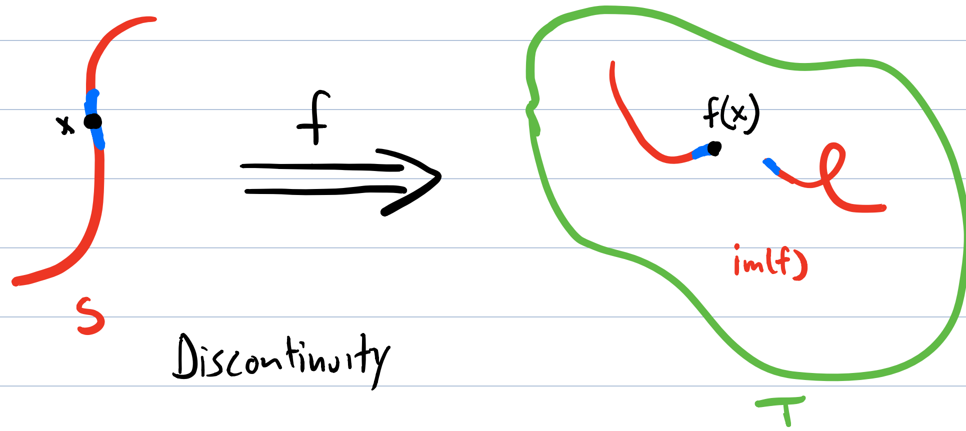
Let $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$ for some $n, m \geq 1$.
Intuitively, a function $f: S \to T$ is continuous if f maps nearby points to nearby points.

d gives us our notion of "nearby."

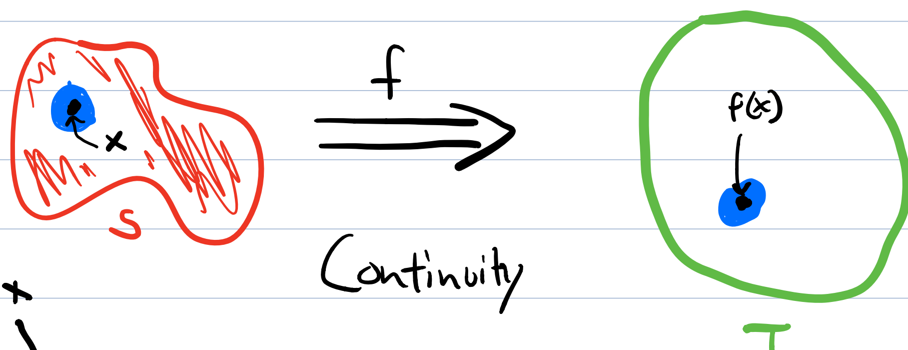
Illustration



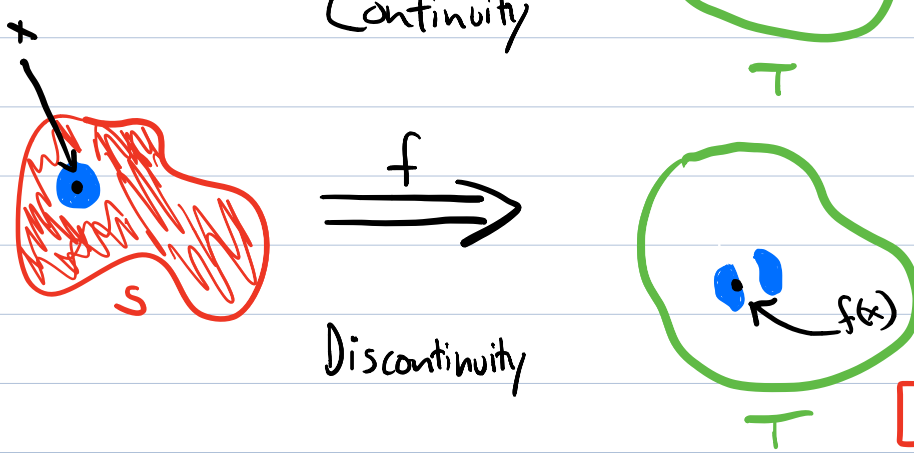
Continuity



Discontinuity



Continuity



Discontinuity

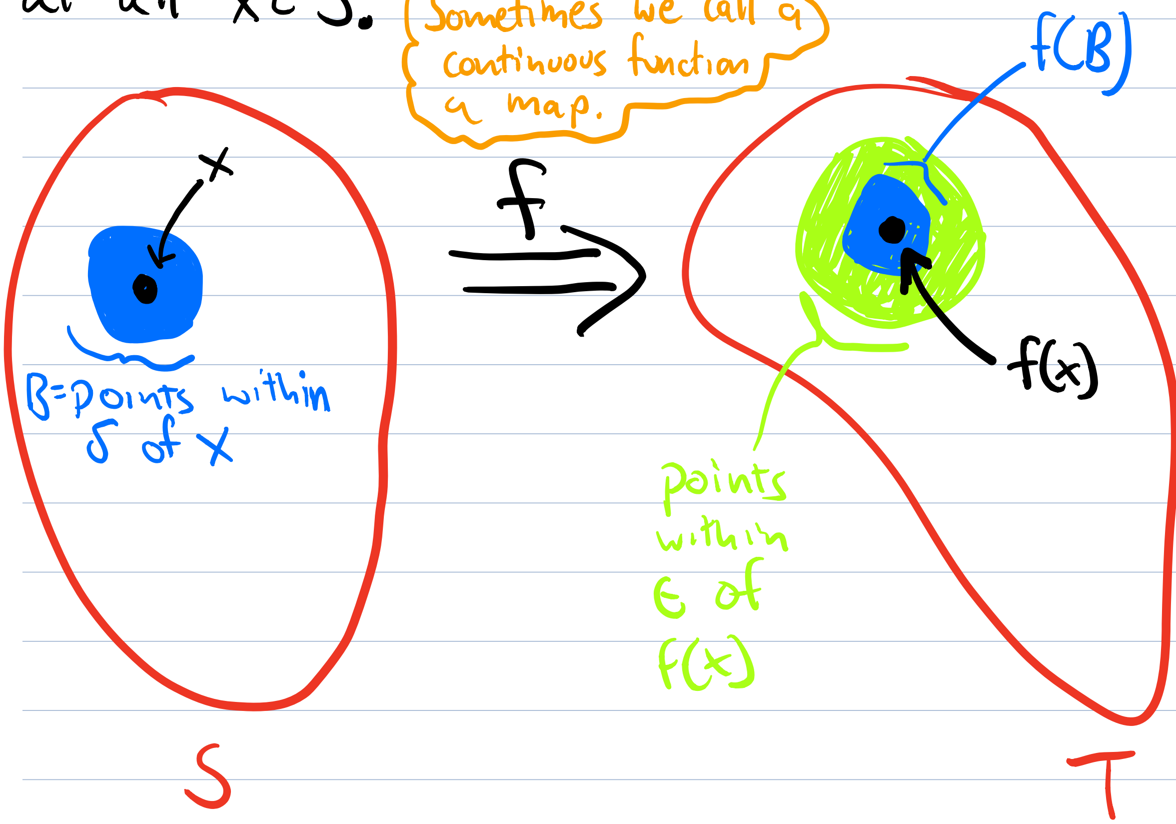
[Lecture ended]

Formal Definition

We say $f: S \to T$ is continuous at $x \in S$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in S$ and $d(x,y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

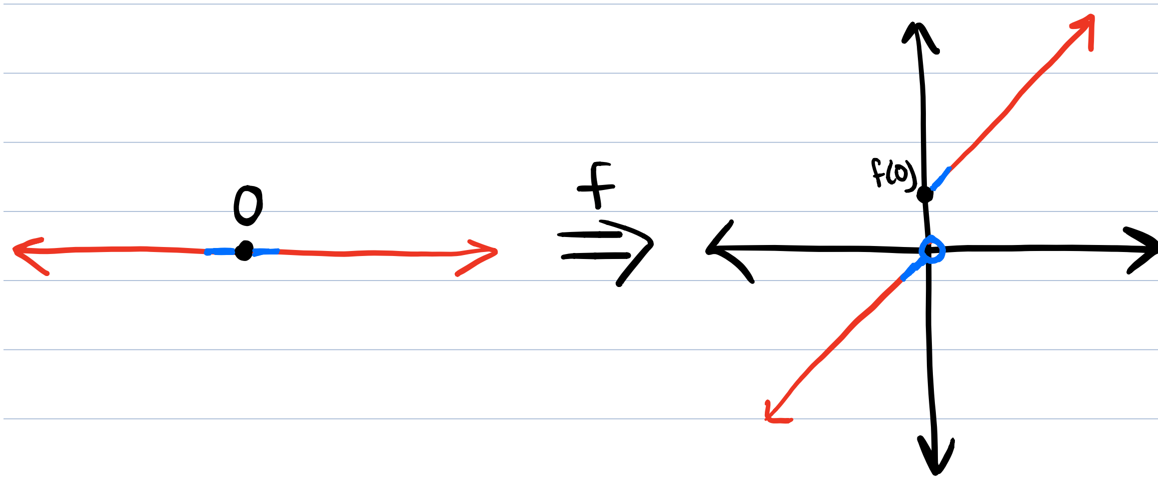
We say f is continuous if it is continuous at all $x \in S$.

Sometimes we call a continuous function a map.



Interpretation: You give me any positive ϵ no matter how small. Continuity at x means that I can choose a positive δ such that points within distance δ of x map under f to points within distance ϵ of x. (I'm allowed to choose δ as small as I want, as long as it's positive.)

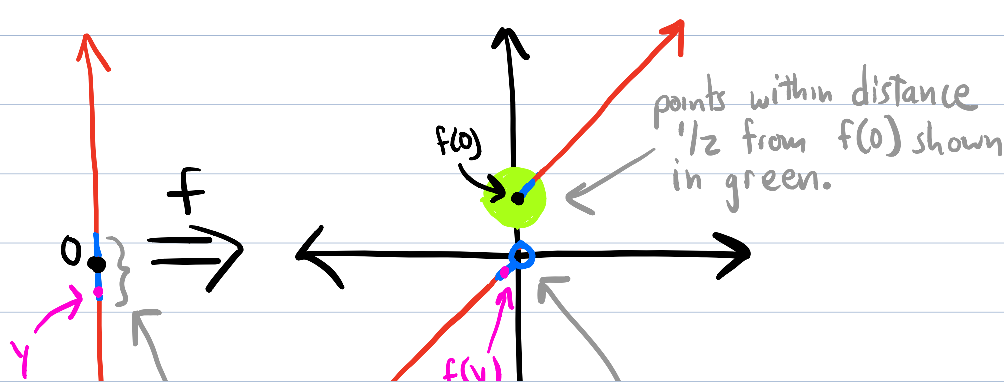
Example: Consider $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\begin{cases}(x,x+1) & \text{if } x<0\\ (x,x) & \text{if } x\geq 0\end{cases}$

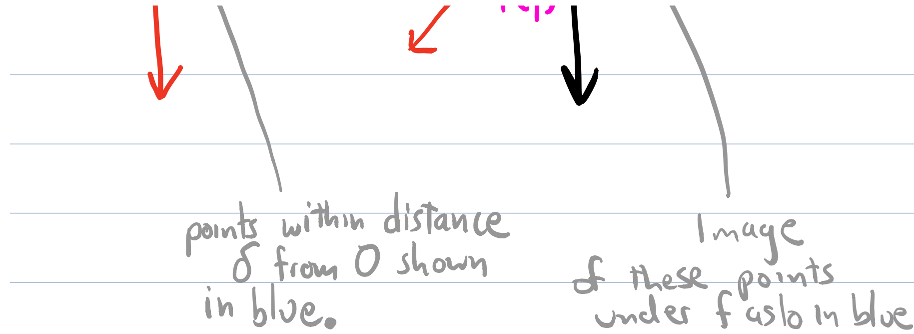


Since f "splits the line" at 0, we expect that f is not continuous. Lets check this using the formal definition of continuity.

Proof that f is not continuous

Let $\epsilon = 1/2$.





No matter how small we take δ, if $y<0$ and $d(0,y)<\delta$, then $d(f(0), f(y)) > \frac{1}{2}$.
Hence f is not continuous at 0.

Examples of continuous functions.

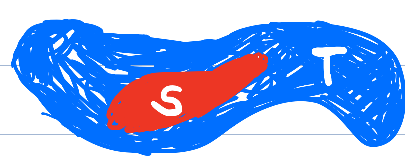
Elementary $\mathbb{R}$-valued functions from calculus are continuous at each point where they are defined, e.g.:

- $\sin x$, $\cos x$, $\log x$, e^x, polynomials
- sums, products, and quotients of these.

4 facts (moral: functions that you think would be continuous usually are).

1) If $f: S \to T$ and $g: T \to U$ are both continuous, then $g \circ f: S \to U$ is continuous.

2) If $S \subset T \subset \mathbb{R}^n$, then the inclusion map $j: S \to T$ given by $j(x) = x$ is continuous.



3) If $U \subset \mathbb{R}^n$ and $f_1, f_2, \ldots, f_n : U \to \mathbb{R}$ are continuous, then $(f_1, f_2, \ldots, f_n): U \to \mathbb{R}^n$, given by $(f_1, f_2, \ldots, f_n)(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ is continuous.

4) If $f: S \to T$ is continuous then the the map $\tilde{f}: S \to \text{im}(f)$ defined by $\tilde{f}(x) = f(x)$ is continuous.

In this class, we won't spend too much time worrying about the rigorous definition of continuity, but I do want you to be familiar with it.

Homeomorphism

For S, T subsets of Euclidean spaces, A function $f: S \to T$ is a homeomorphism if

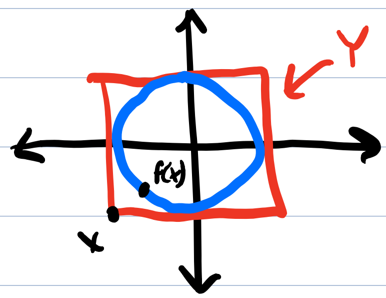
1) f is a continuous bijection ← bijection = has inverse
2) The inverse of f is also continuous.

Homeomorphism is the main notion of continuous deformation we'll consider in this course.

If $\exists$ a homeomorphism $f: S \to T$, we say S and T are homeomorphic.

In This class, "topologically equivalent" = homeomorphic.

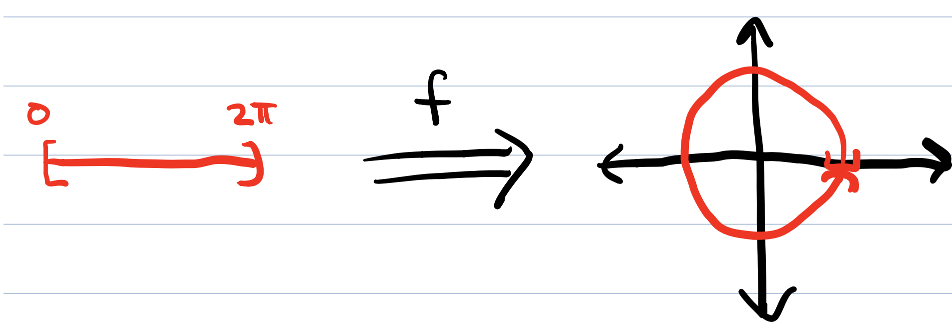
Example Let $Y \subset \mathbb{R}^2$ be the square of side length 2, embedded in the plane as shown



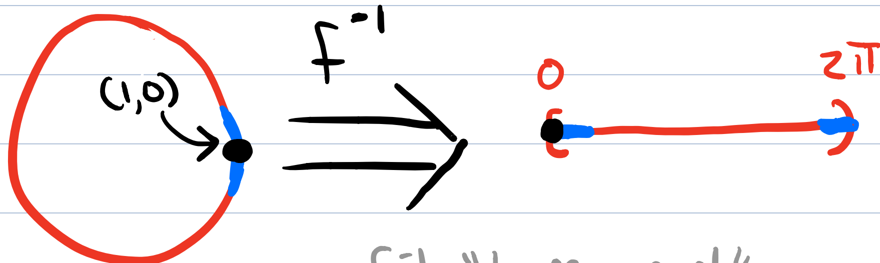
The function $f: Y \to S^1$ given by
$f(x) = \frac{x}{\|x\|}$ is a homeomorphism.
where $\|x\|$ = distance of x to origin
$= \sqrt{x_1^2 + x_2^2}$

By facts above, this is continuous.
It is intuitively clear that this is a bijection with a continuous inverse. The inverse can be written down, but we won't bother.

Example: Consider the function
$f: [0, 2\pi) \to S^1$ from last lecture
given by $f(x) = (\cos x, \sin x)$.



f is continuous, and we saw last lecture that it is a bijection. However, $f^{-1}: S^1 \to [0, 2\pi)$ is not continuous at $(1,0)$. (And therefore, f is not continuous.)



f^{-1} "tears apart" any small neighborhood around $(1,0)$.

<u>Note</u>: The fact that f is not a homeomorphism doesn't imply that $[0, 2\pi)$ and S^1 are not homeomorphic. In fact they are not, and we will explain why soon.

<u>Basic Facts About Homeomorphisms.</u>

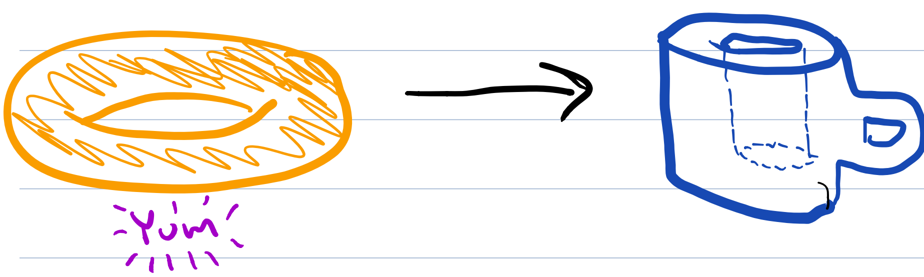
- Clearly, if $f: S \to T$ a homeomorphism, then f^{-1} is a homeomorphism.

- If $f: S \to T$ and $g: T \to U$ are homeomorphisms, then $g \circ f: S \to T$ is a homeomorphism (w/ inverse $f^{-1} \circ g^{-1}$)

Example: Returning to examples from the 1st day of class, consider the capital letters as unions of curves (no thickness)

D and O are homeomorphic
T, Y, and J, E, and F are G homeomorphic
C, S, and Z homemorphic.
X and K are homeomorphic (at least, the way I write K.)

Example: The donut and coffee mug are homeomorphic



Isotopy

All of the pair of homeomorphic spaces we've seen so far are topologically equivalent in a sense that's stronger than homeomorphism, called isotopy.