AMAT 583 Lecture 5, 9/10/19

Last lecture: Informal discussion of continuity

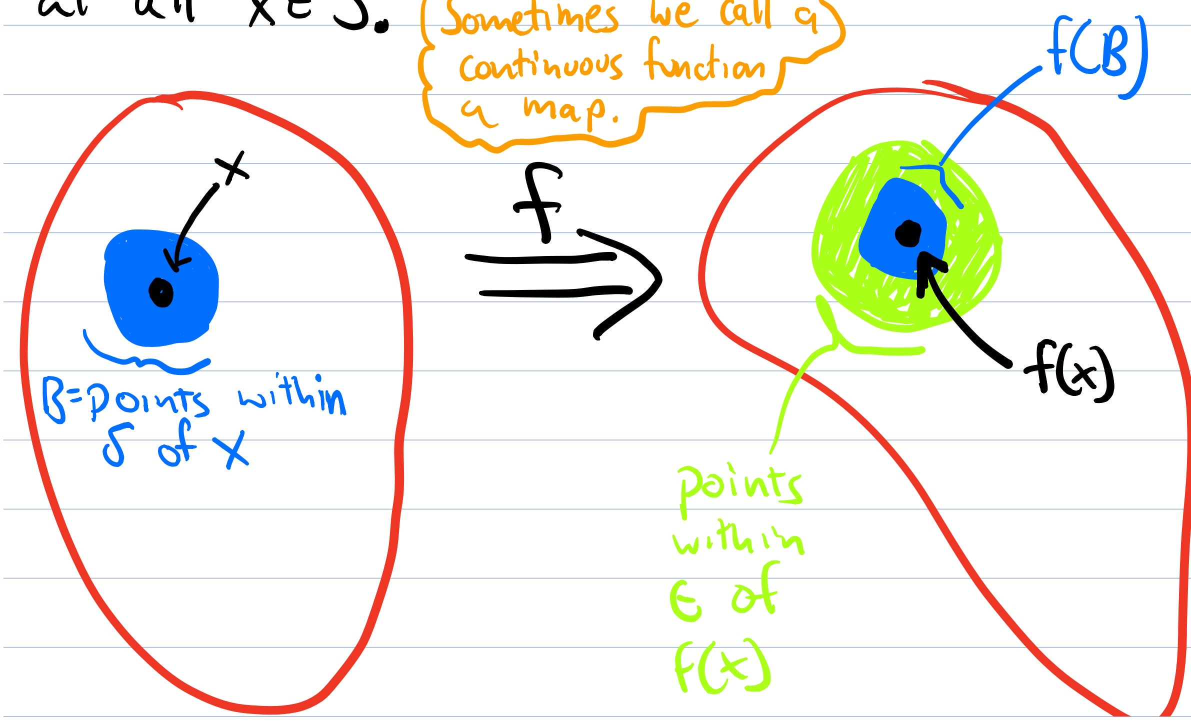
Today:
- Formal definition of continuity
- Properties of continuous functions
- Homeomorphism

Formal Definition of Continuity

We say $f: S \to T$ is continuous at $x \in S$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in S$ and $d(x,y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

We say f is continuous if it is continuous at all $x \in S$.

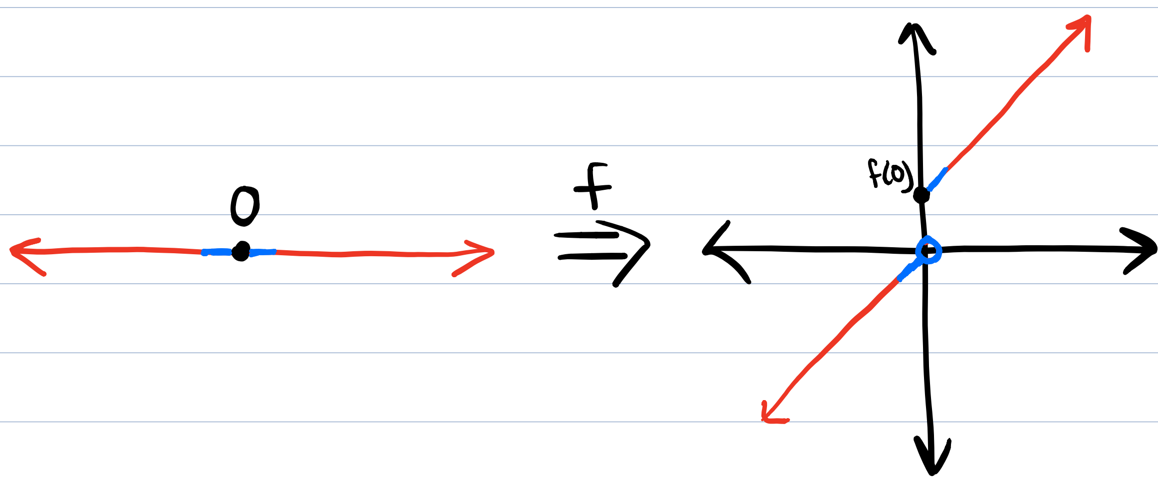
Sometimes we call a continuous function a map.



S Illustration of continuity at x_0 $\tilde{T}$

Interpretation: You give me any positive ϵ no matter how small. Continuity at x means that I can choose a positive δ such that points within distance δ of x map under f to points within distance ϵ of $f(x)$. (I'm allowed to choose δ as small as I want, as long as it's positive.)

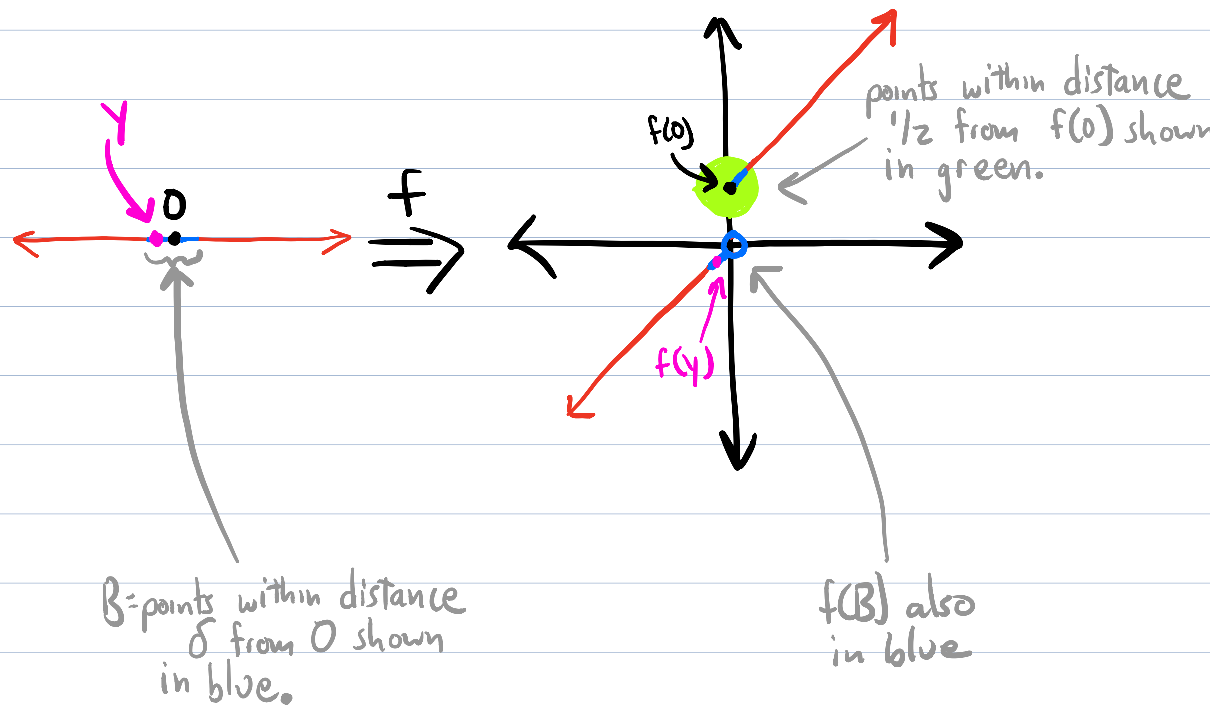
Example: Consider $f : \mathbb{R} \to \mathbb{R}^2$ defined by
$$f(x) = \begin{cases} (x, x+1) & \text{if } x < 0 \\ (x, x) & \text{if } x \geq 0 \end{cases}$$



Since f "splits the line" at 0, we expect that f is not continuous. Lets check this using the formal definition of continuity.

Proof that f is not continuous

Let $\epsilon = 1/2$.



For all $y < 0$, $d(f(y), f(0)) = d((y,y),(1,0)) > \frac{1}{2}$

No matter how small we take δ, there is always some $y < 0$ with $d(0,y) < \delta$. Such y doesn't satisfy $d(f(0), f(y)) < \frac{1}{2}$.

That is, y doesn't map into the green disk.

Hence f is not continuous at 0.

Examples of continuous functions.

Elementary $\mathbb{R}$-valued functions from calculus are continuous at each point where they are defined, e.g.:

- $\sin x$, $\cos x$, $\log x$, e^x, polynomials
- sums, products, and quotients of these.

4 facts

(moral: functions that you think would be continuous usually are).

1) If $f: S \to T$ and $g: T \to U$ are both continuous, then $g \circ f: S \to U$ is continuous. Ex: $f(x) = x^2$, $g(x) = \sin x$, $g \circ f = \sin x^2$.

2) If $S \subset T \subset \mathbb{R}^n$, then the inclusion map $j: S \to T$ given by $j(x) = x$ is continuous.

3) If $U \subset \mathbb{R}^m$ and $f_1, f_2, \ldots, f_n: U \to \mathbb{R}$ are continuous, then $(f_1, f_2, \ldots, f_n): U \to \mathbb{R}^n$, given by $(f_1, f_2, \ldots, f_n)(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ is continuous. Ex: $f_1: \mathbb{R} \to \mathbb{R}$, $f_1(x) = \cos x$; $f_2: \mathbb{R} \to \mathbb{R}$, $f_2(x) = \sin x$, $(f_1, f_2) = (\cos x, \sin x)$ is continuous.

4) If $f: S \to T$ is continuous then the the map $\tilde{f}: S \to \operatorname{im}(f)$ defined by $\tilde{f}(x) = f(x)$ is continuous.

Ex: $f(x): \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is continuous $\Rightarrow \tilde{f}(x): \mathbb{R} \to [0, \infty)$,

$\tilde{f}(x) = x^2$ is continuous

In this class, we won't spend too much time worrying about the rigorous definition of continuity, but I do want you to be familiar with it.

Homeomorphism

For S, T subsets of Euclidean spaces,
A function $f: S \to T$ is a <u>homeomorphism</u> if

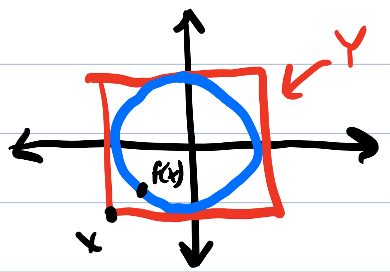
1) f is a continuous bijection ← bijection = has inverse
2) The inverse of f is also continuous.

Homeomorphism is one of the main notions of continuous deformation we'll consider in this course.

If $\exists$ a homeomorphism $f: S \to T$, we say S and T are <u>homeomorphic</u>.

<u>Intuition</u>: f is a bijection such that neither f nor f^{-1} tears its domain.

<u>Example</u> Let $Y \subset \mathbb{R}^2$ be the square of side length 2, embedded in the plane as shown

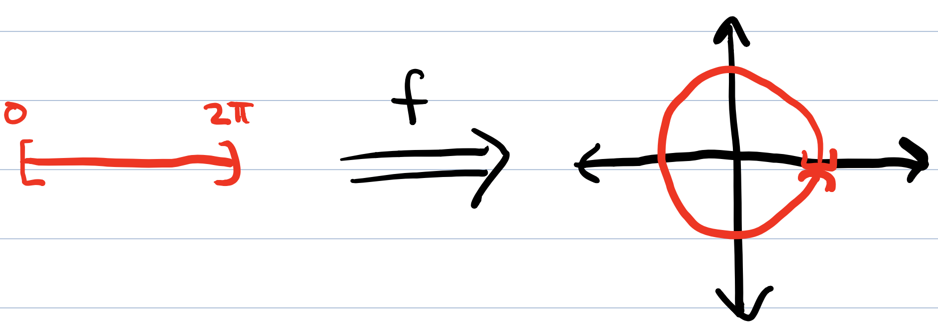


The function $f: Y \to S^1$ given by
$f(x) = \frac{x}{\|x\|}$ is a homeomorphism.
where $\|x\|$ = distance of x to origin
$= \sqrt{x_1^2 + x_2^2}$

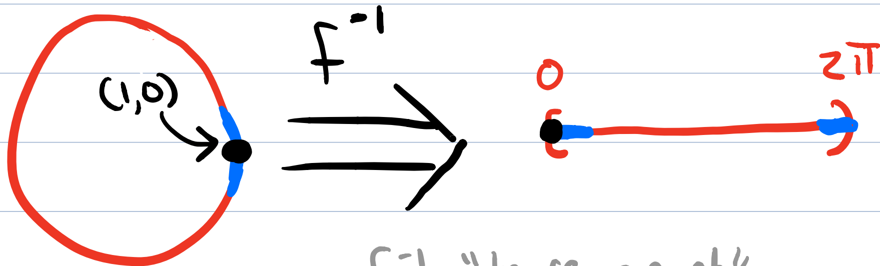
By facts above, this is continuous.
It is intuitively clear that this is a bijection with a continuous inverse. The inverse can be written down, but we won't bother.

Note: When we talked about continuous deformations on the first day of class, thinking of objects made of rubber, there was an implicit notion of an object evolving in time from an undeformed state to a deformed state. However, the definition of homeomorphism does not model any such temporal dynamics. We will return to this point soon.

Example: Consider the function
$f: [0, 2\pi) \longrightarrow S^1$ from last lecture
given by $f(x) = (\cos x, \sin x)$.



f is continuous, and we saw last lecture that it is a bijection. However, $f^{-1}: S^1 \to [0, 2\pi)$ is not continuous at $(1,0)$. (And therefore, f is not continuous.)



f^{-1} "tears apart" any small neighborhood around $(1,0)$.

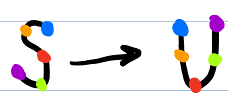
Note: The fact that f is not a homeomorphism doesn't imply that $[0, 2\pi)$ and S^1 are not homeomorphic. In fact, they are not, and we will explain why later in the course.

__Example__: Consider the capital letters as unions of curves in the plane with no thickness.

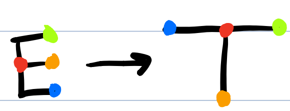
T is homeomorphic to Y:

for example, one can define a homeomorphism $T \to Y$ which sends each of the colored points of T above to the point of Y of the same color.

S is homeomorphic to U:



E is homeomorphic to T:



O is not homeomorphic to S. Intuitively, any bijection $O \to S$ must "cut the O" somewhere, so cannot be continuous.

Note: In general, subsets of $\mathbb{R}^2$ with different "#'s of holes" are not homeomorphic. (Making this formal requires ideas from algebraic topology that we will discuss later in the course.

Example: B is not homeomorphic to any other letter, because B is the only capital letter with two holes.

Example X is not homeomorphic to Y.

Explanation: X has a point where 4 line segments meet, Y does not. Using this, one can show that X and Y are not homeomorphic.

Basic Facts About Homeomorphisms.

- Clearly, if $f: S \to T$ a homeomorphism, then f^{-1} is a homeomorphism.

- If $f: S \to T$ and $g: T \to U$ are homeomorphisms, then $g \circ f: S \to T$ is a homeomorphism (w/ inverse $f^{-1} \circ g^{-1}$)

as an immediate consequence, if X and Y are homeomorphic, and Y and Z are homeomorphic, then X and Z are homeomorphic.