AMAT 583, Sept. 12 (Lec. 6)

Today: homeomorphism, continued
isotopy

Recall the following key definition from last time:

Homeomorphism

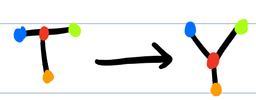
For S, T subsets of Euclidean spaces,
A function $f: S \to T$ is a **homeomorphism** if

1) f is a continuous bijection
2) The inverse of f is also continuous.

Last lecture, we looked at some examples illustrating this definition. We now consider several more.

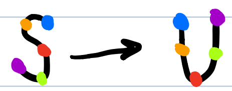
Example: Consider the capital letters as unions of curves in the plane with no thickness.

T is homeomorphic to Y:

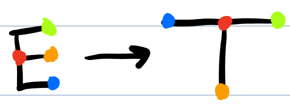


for example, one can define a homeomorphism $T \to Y$ which sends each of the colored points of T above to the point of Y of the same color.

S is homeomorphic to U:



E is homeomorphic to T:



O is not homeomorphic to S. Intuitively, any bijection $O \to S$ must "cut the O" somewhere, so cannot be continuous.

<u>Note:</u> In general, subsets of $\mathbb{R}^2$ with different "#'s of holes" are not homeomorphic. (Making this formal requires ideas from algebraic topology that we will discuss later in the course.

<u>Example:</u> B is not homeomorphic to any other letter, because B is the only capital letter with two holes.

<u>Example</u> X is not homeomorphic to Y.
<u>Explanation:</u> X has a point where 4 line segments meet, Y does not. Using this, one can show that X and Y are not homeomorphic.

Basic Facts About Homeomorphisms.

- Clearly, if $f: S \to T$ a homeomorphism, then f^{-1} is a homeomorphism.

- If $f: S \to T$ and $g: T \to U$ are homeomorphisms, then $g \circ f: S \to T$ is a homeomorphism (w/ inverse $f^{-1} \circ g^{-1}$)

  as an immediate consequence, if X and Y are homeomorphic, and Y and Z are homeomorphic, then X and Z are homeomorphic.

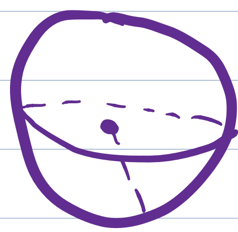
Isotopy

All of the pair of homeomorphic spaces we've seen so far are topologically equivalent in a sense that's stronger than homeomorphism, called isotopy.

The definition of isotopy is closer to the "rubber-sheet geometry" idea of continuous deformation that we introduced on the first day.

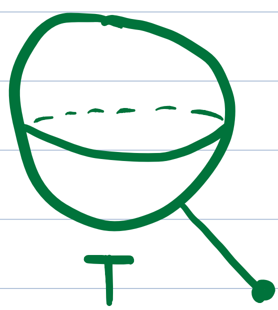
Motivating example

Let $S, T \subset \mathbb{R}^3$ be as illustrated:



S

S is a unit circle with a line segment attached to one point. The line segment points inward.



T

T is also a unit circle with a line segment attached to the same point, but now line segment points outward.

S and T are homeomorphic.

However, if S and T were made of rubber, we couldn't deform S into T without tearing. The line segment would have to pass through the sphere.

Formally, we express this idea using <u>isotopy</u>.

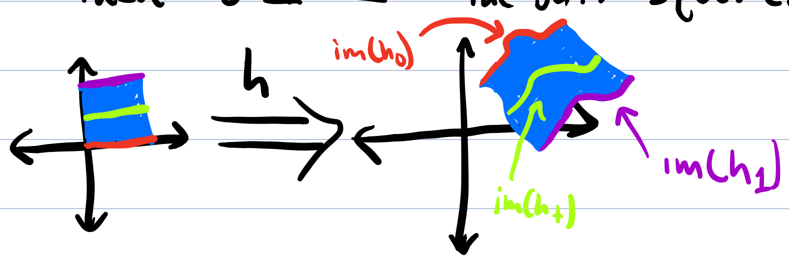
To define isotopy, we need to first define <u>homotopies</u> and <u>embeddings</u>

Homotopy is a notion of continuous deformation for <u>functions</u> (rather than spaces).

For $S \subset \mathbb{R}^n$, $h : \overbrace{S \times I}^{\text{thickening of } S} \to T$ a continuous function and $t \in I$, let $h_t : S \to T$ be given by
$h_t(x) = h(x,t)$.

<u>Interpretation</u>: we can think of h as a family of continuous functions $\{h_t \mid t \in I\}$ from S to T evolving in time. (We interpret t as time.) The continuity of h means that h_t "evolves continuously" as t changes.

<u>Example</u>: $S = I$, $T = \mathbb{R}^2$.
Then $S \times I = I^2$ = the unit square.



Each $h_t : I \to \mathbb{R}^2$ species a curve in $\mathbb{R}^2$.
As t increases, these curves evolve continuously

<u>Definition</u>: For continuous maps $f, g : S \to T$ a <u>homotopy</u> from f to g is a continuous map

$h: S \times I \to T$

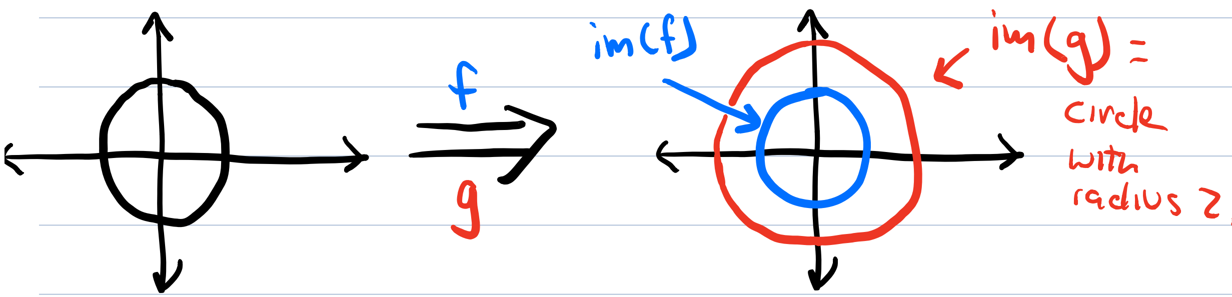
such that $h_0 = f$ and $h_1 = g$.

Note: Any continuous map $h: S \times I \to T$ is a homotopy from h_0 to h_1.

We sometimes call h a homotopy without mentioning h_0, h_1.

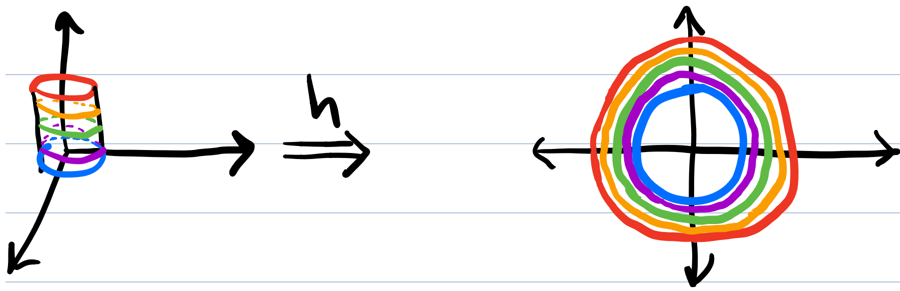
Example $f, g: S^1 \to \mathbb{R}^2$ $\quad f(\vec{x}) = \vec{x}$ (f is the inclusion map.)

unit circle in $\mathbb{R}^2$. $\quad g(\vec{x}) = 2\vec{x}$



Let $h: S^1 \times I \to \mathbb{R}^2$ be given by
$h(x, t) = (1+t)\vec{x}$.
Then $h_t: S^1 \to \mathbb{R}^2$ is given by $h_t(x) = (1+t)\vec{x}$,
and clearly $h_0 = f$, $h_1 = g$.

$S^1 \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$, so $S^1 \times I \subset \mathbb{R}^3$.
In fact, $S^1 \times I$ is a cylinder, and the following illustrates h:



$\text{im}(h_t)$ is shown above for $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

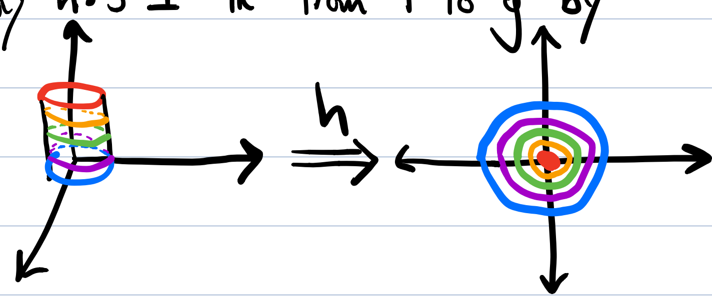
[Lecture ended here]

Example: Let $f : S^1 \to \mathbb{R}^2$ be the inclusion map, and

This example is similar to the last one and will be skipped in class.

let
$$g : S^1 \to \mathbb{R}^2 \text{ be given by}$$
$$g(x) = (0,0) \text{ for all } x \in S^1.$$

We specify a homotopy $h : S^1 \times I \to \mathbb{R}^2$ from f to g by
$h(\vec{x}, t) = t\vec{x}$.



Note that $\text{im}(h_t)$ is a circle for $t < 1$ and a point for $t = 1$. as above, $\text{im}(h_t)$ is shown for $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

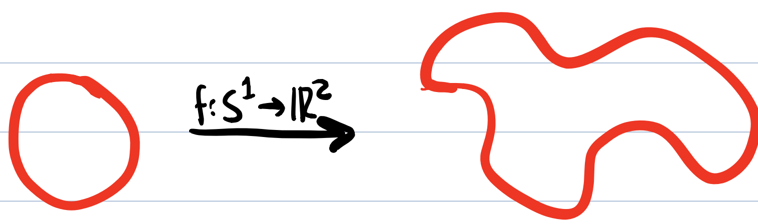
Embeddings

Recall: for any function $f : S \to T$, there is an associated function onto the image of f, namely

$$\tilde{f} : S \to \mathrm{im}(f)$$

given by $\tilde{f}(x) = f(x)$. That is f and $\tilde{f}$ are given by the same rule, but the codomain of $\tilde{f}$ is as small as possible.

Def: A continuous map $f : S \to T$ is an embedding if f is a $\underbrace{\text{homeomorphism onto its image}}_{\text{i.e., } \tilde{f} \text{ is a homeomorphism}}$. (for concreteness, think of T as $\mathbb{R}^n$)



embedding

not an embedding

Fact: Any embedding is an injection but not every continuous injection is an embedding.

Proof of injectivity: If $\tilde{f}$ is a homeomorphism then it is bijective, hence injective. $f = j \circ \tilde{f}$, where $j: \mathrm{im}(f) \to T$ is the inclusion map. j is injective. The composition of two injective functions is injective, so f is injective. $\blacksquare$

Example: The following illustrates that a continuous injection is not necessarily an embedding

Consider $f: [0, 2\pi) \to \mathbb{R}^2$, $f(x) = (\cos x, \sin x)$. We seen above that $\tilde{f}$ is a continuous bijection but not a homeomorphism.

Isotopy

Definition: For $S, T \subset \mathbb{R}^n$ an isotopy from S to T is a homotopy $h: X \times I \to \mathbb{R}^n$ such that

$\mathrm{im}(h_0) = S$, $\mathrm{im}(h_1) = T$,

$h_t: X \to \mathbb{R}^n$ is an embedding for all $t \in I$.

If there exists an isotopy from S to T, we say S and T are isotopic. Note: It follows from the definition that X is homeomorphic to both S and T.

Interpretation:
- $\mathrm{im}(h_t)$ is the snapshot at time t of a continuous deformation from S to T.
- Continuity of h ensures that these "snapshots"

evolve continuously in time.

- fact that h_t is an embedding ensures all $\operatorname{im}(h_t)$ are homeomorphic.

Example: Let $T \subset \mathbb{R}^2$ be the circle of radius 2 centered at the origin.

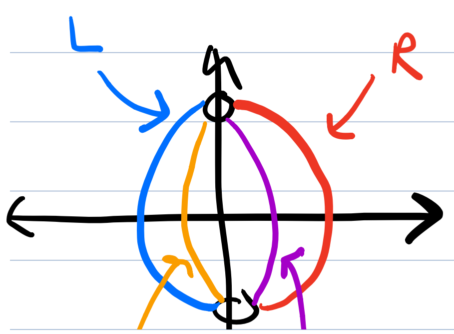
The homotopy $h: S^1 \times I \to \mathbb{R}^2$, $h(\vec{x},t) = (1+t)\vec{x}$ in the example above is an isotopy from S^1 to T. ← circle of radius 2.

Note: If S and T are isotopic, then they are homeomorphic; for h any isotopy from S to T, $\tilde{h}_1 \circ \tilde{h}_0^{-1}$ is a homeomorphism from S to T.

Explanation: $h_1 : S \to \mathbb{R}^n$ is an embedding, hence a homeomorphism onto its image. But $\operatorname{im}(h_1) = T$.

Example: Let $L = \{(x,y) \in S^1 \mid x < 0\}$

$R = \{(x,y) \in S^1 \mid x > 0\}$.



$h: L \times I \to \mathbb{R}^2$,

$h((x,y),t) = ((1-2t)x, y)$

$\underset{\text{im } h_{1/4}}{(}$ $\downarrow$ $\underset{\text{im } h_{3/4}}{\backslash}$ is a homotopy from L to R.

Explanation: $h_0(x,y) = ((1-0)x, y) = (x,y)$ so $h_0 = Id_L$.

$h_1(x,y) = (-x, y)$, so $\text{im } h_1 = R$.

$h_t(x,y) = ((1-2t)x, y)$.

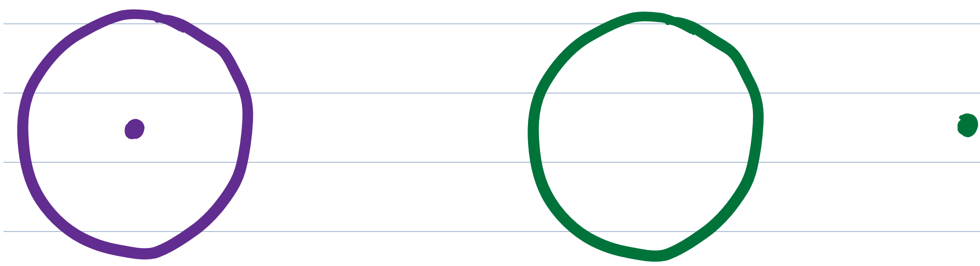
Not hard to check that each h_t is an embedding.

Example:

S and T are not isotopic.

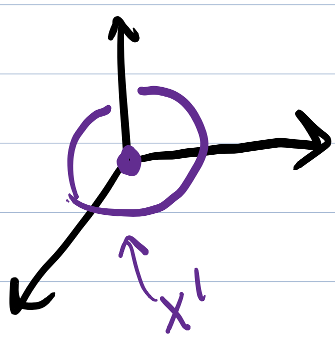
Note: Whether S and T are isotopic depends on where S and T are embedded. (That's not true for homeomorphism!)

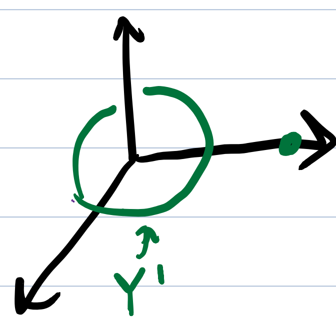
Example $X = S^1 \cup \{0\} \subset \mathbb{R}^2$ $\quad Y = S^1 \cup \{(3,0)\}$.

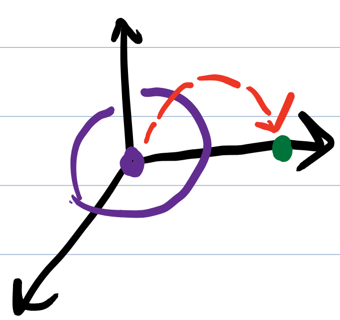


X and Y homeomorphic, not isotopic.
But if we embed X, Y in $\mathbb{R}^3$, then they are isotopic there.

That is, let $X' = \{(x,y,0) \mid (x,y) \in X\} \subset \mathbb{R}^3$
$Y' = \{(x,y,0) \mid (x,y) \in Y\} \subset \mathbb{R}^3$







There's an isotopy $h: X' \times I \to \mathbb{R}^3$ which moves the extra point as shown in red.

Similarly, if we embed S and T of the previous example into $\mathbb{R}^4$, they are isotopic there.

Facts about isotopies: The same properties of

Symmetry: If there exists an isotopy from S to T, then there exists an isotopy from T to S.

"Isotopies can be reversed"

Pf: If $h: X \times I \to \mathbb{R}^n$ is an isotopy from S to T then $\bar{h}: X \times I \to T$, given by $\bar{h}(x,t) = h(x, 1-t)$ is an isotopy from T to S.

Transitivity: If S, T are isotopic and T, U are isotopic, so are S, U.

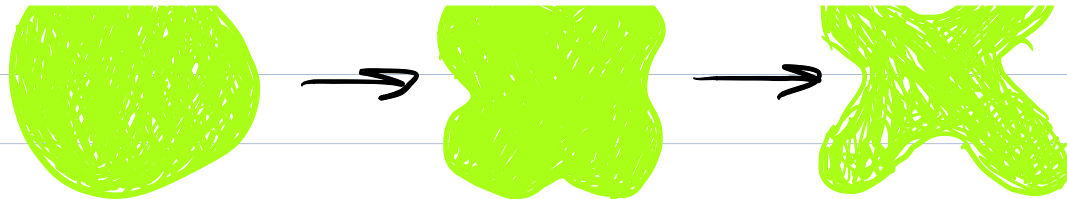
(The proof takes just a few lines.)

Example: Consider the thick capital letters

X Y

Both are isotopic to the disc $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Example:



Isotopy from D to **X**

Hence, by transitivity, **X** and **Y** are isotopic.
In particular, they are homeomorphic.

Thus we see that whether two letters are homeomorphic depends on whether we consider the thin or thick versions.

Later,