AMAT 583 Lec 7, 9/17/19

Today: Homotopy continued
Embeddings
Isotopy

Definition: For continuous maps $f, g: S \to T$ a **homotopy** from f to g is a continuous map

$$h: S \times I \to T$$

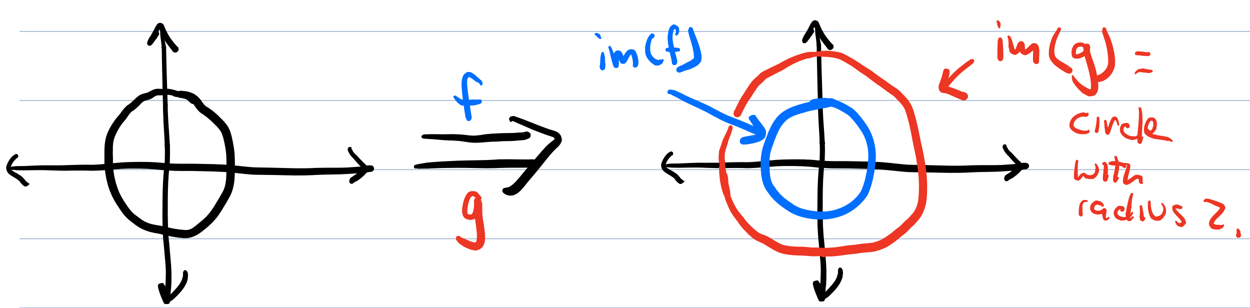
such that $h_0 = f$ and $h_1 = g$.

Note: Any continuous map $h: S \times I \to T$ is a homotopy from h_0 to h_1.

We sometimes call h a homotopy without mentioning h_0, h_1.

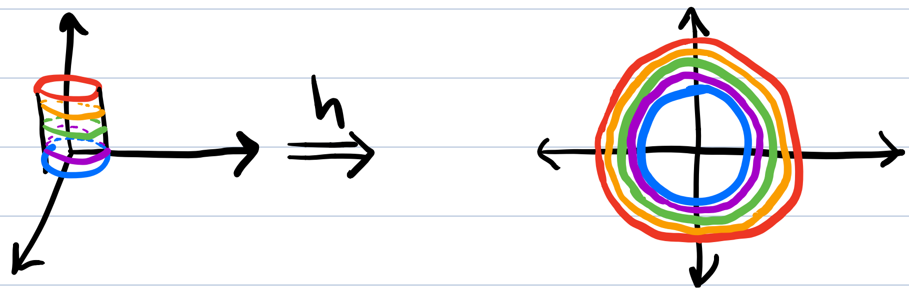
review

Example $f, g: S^1 \to \mathbb{R}^2$ $\quad f(\vec{x}) = \vec{x}$ (f is the inclusion map.)
S^1: unit circle in $\mathbb{R}^2$. $\quad g(\vec{x}) = 2\vec{x}$



Let $h: S^1 \times I \to \mathbb{R}^2$ be given by
$h(\vec{x}, t) = (1+t)\vec{x}$.
Then $h_t: S^1 \to \mathbb{R}^2$ is given by $h_t(\vec{x}) = (1+t)\vec{x}$,
and clearly $h_0 = f$, $h_1 = g$.

$S^1 \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$, so $S^1 \times I \subset \mathbb{R}^3$.
In fact, $S^1 \times I$ is a cylinder, and the following illustrates h:



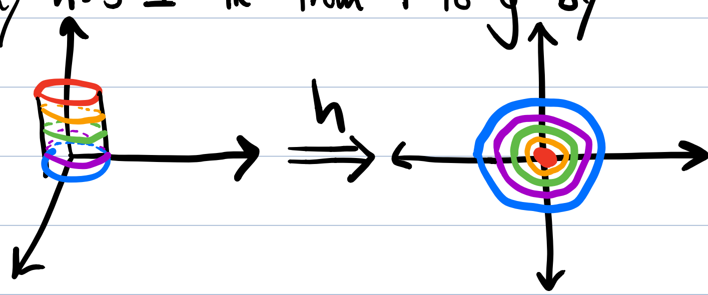
$\mathrm{im}(h_t)$ is shown above for $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

Example: Let $f: S^1 \to \mathbb{R}^2$ be the inclusion map, and

This example is similar to the last one and will be skipped in class.

let $g: S^1 \to \mathbb{R}^2$ be given by
$g(x) = (0,0)$ for all $x \in S^1$.

We specify a homotopy $h: S^1 \times I \to \mathbb{R}^2$ from f to g by
$h(\vec{x}, t) = t\vec{x}$.



Note that $\mathrm{im}(h_t)$ is a circle for $t<1$ and a point for $t=1$. as above, $\mathrm{im}(h_t)$ is shown for $t=0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

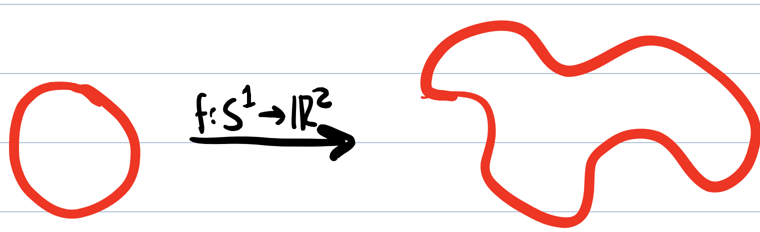
Embeddings

Recall: for any function $f: S \to T$, there is an associated function onto the image of f, namely

$$\tilde{f}: S \to \mathrm{im}(f)$$

given by $\tilde{f}(x) = f(x)$. That is f and $\tilde{f}$ are given by the same rule, but the codomain of $\tilde{f}$ is as small as possible.

Def: A continuous map $f: S \to T$ is an embedding if f is a $\underbrace{\text{homeomorphism onto its image}}_{\text{i.e., } \tilde{f} \text{ is a homeomorphism}}$. (for concreteness, think of T as $\mathbb{R}^n$)



embedding

not an embedding

Fact: Any embedding is an injection but not every continuous injection is an embedding.

Proof of injectivity: If $\tilde{f}$ is a homeomorphism then it is bijective, hence injective. $f = j \circ \tilde{f}$, where $j : \mathrm{im}(f) \to T$ is the inclusion map. j is injective. The composition of two injective functions is injective, so f is injective. $\blacksquare$

Example: The following illustrates that a continuous injection is not necessarily an embedding

Consider $f : [0, 2\pi) \to \mathbb{R}^2$, $f(x) = (\cos x, \sin x)$.
We seen above that $\tilde{f}$ is a continuous bijection but not a homeomorphism.

Remark: If S has a property called compactness, then any continuous injection $f: S \to \mathbb{R}^n$ is an embedding.

e.g., S^1 is compact, but $[0, 2\pi)$ is not compact.

Isotopy

Definition: For $S, T \subset \mathbb{R}^n$ an *isotopy* from S to T is a homotopy $h: X \times I \to \mathbb{R}^n$ such that

$$\mathrm{im}(h_0) = S, \quad \mathrm{im}(h_1) = T,$$

$$h_t : X \to \mathbb{R}^n \text{ is an embedding for all } t \in I.$$

If there exists an isotopy from S to T, we say S and T are *isotopic*.

Interpretation: - $\mathrm{im}(h_t)$ is the snapshot at time t of a continuous deformation from S to T.

- Continuity of h ensures that these "snapshots" evolve continuously in time.
- fact that h_t is an embedding ensures all $\mathrm{im}(h_t)$ are homeomorphisms.

Clarifications:

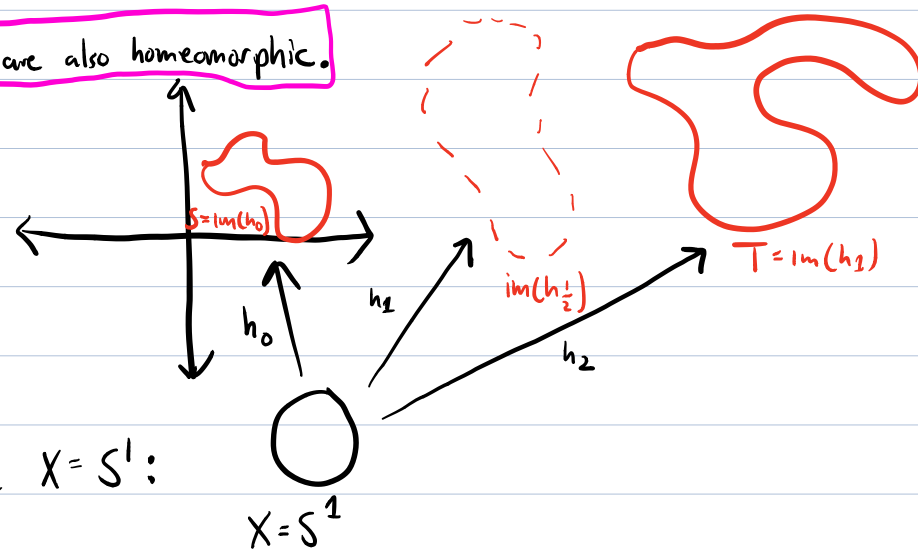
- In the above definition, homotopy just means "continous map."
- The definition doesn't explictly put any requirements on X, but it follows from the defintion that X has to be homeomorphic to both S and T:

  $\tilde{h}_0 : X \to \underset{S}{\underset{\parallel}{\mathrm{im}(h_0)}}$ is a homeomorphism, because h_0 is an embedding.

$\tilde{h}_1 : X \to \underbrace{\operatorname{im}(h_1)}_{= T}$ is a homeomorphism because h_1 is an embedding.

$\boxed{\text{hence, } S \text{ and } T \text{ are also homeomorphic.}}$

<u>Illustration:</u>

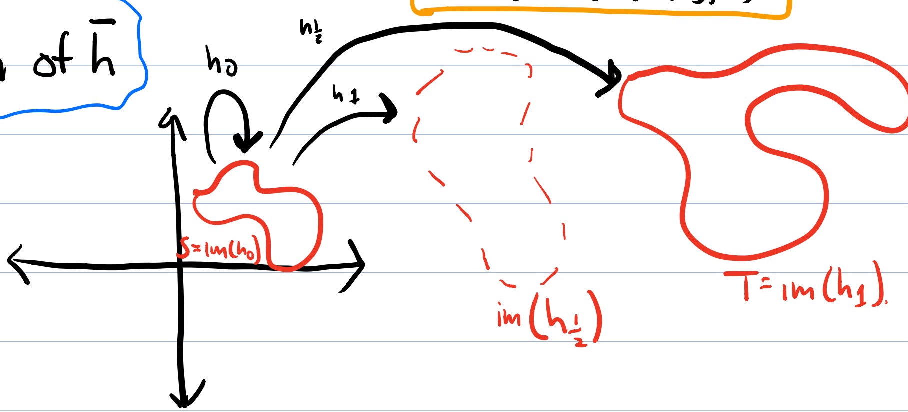


I can take $X = S^1$:

<u>Fact:</u> If there exists an isotopy h from S to T then there exists an isotopy $\bar{h}: S \times I \to \mathbb{R}^n$ (that is, $X = S$!) with $h_0 : S \hookrightarrow \mathbb{R}^n$ the inclusion.

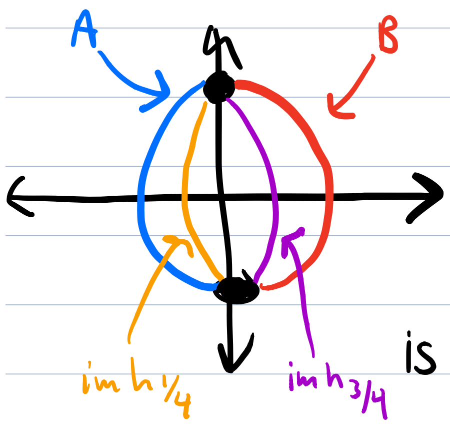
<u>Proof</u>: define $\bar{h}$ by $\bar{h}(x,t) = h(\tilde{h}_0^{-1}(x), t)$.

Illustration of $\bar{h}$



Example: Let $A = \{(x,y) \in S^1 \mid x \leq 0\}$

$B = \{(x,y) \in S^1 \mid x \geq 0\}$.



$h: L \times I \to \mathbb{R}^2$,

$$h((x,y), t) = ((1-2t)x, y)$$

is an isotopy from A to B.

Explanation: $h_0(x,y) = ((1-0)x, y) = (x,y)$ so h_0 is the inclusion of A into $\mathbb{R}^2$.

$h_1(x,y) = (-x, y)$, so $\operatorname{im} h_1 = B$.

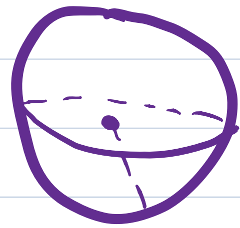
$h_t(x,y) = ((1-2t)x, y)$.

Not hard to check that each h_t is an embedding.

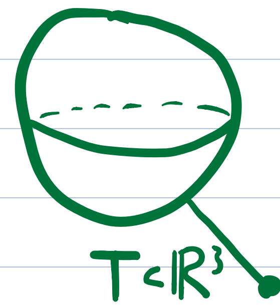
Example: Let $T \subset \mathbb{R}^2$ be the circle of radius 2 centered at the origin.

The homotopy $h: S^1 \times I \to \mathbb{R}^2$, $h(\vec{x}, t) = (1+t)\vec{x}$ in the example above is an isotopy from S^1 to T. circle of radius 2.

Example:



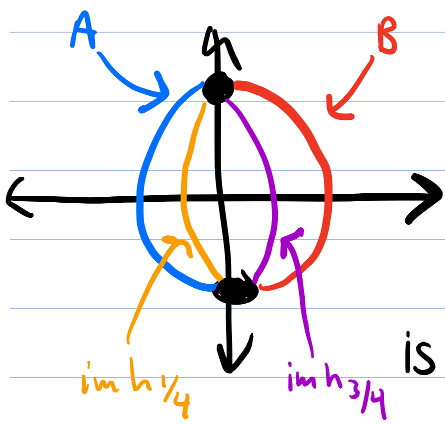
$S\subset\mathbb{R}^3$



$T\subset\mathbb{R}^3$

Example: Let $A=\{(x,y)\in S^1 \mid x\leq 0\}$

$B=\{(x,y)\in S^1 \mid x\geq 0\}$.



$h: L\times I\to\mathbb{R}^2$,

$$h((x,y),t)=((1-2t)x,y)$$

is an isotopy from A to B.

Explanation: $h_0(x,y)=((1-0)x,y)=(x,y)$ so h_0 is the inclusion of A into $\mathbb{R}^2$.

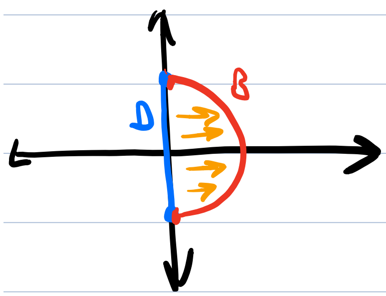
$h_1(x,y)=(-x,y)$, so $\operatorname{im} h_1=B$.

$h_t(x,y)=((1-2t)x,y)$.

Not hard to check that each h_t is an embedding.

Exercise: Let B be as in the last example.
(will not be covered in class)
Let $D = \{0\} \times [-1,1] = \{(0,y) \mid -1 \leq y \leq 1\}$.

a) Give a homeomorphism $f: D \to B$.



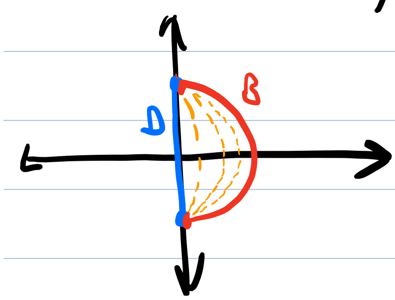
Answer: $f(0,y) = (\sqrt{1-y^2}, y)$

note: $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$
$f(x,y) \in S^1$ because
$(\sqrt{1-y^2})^2 + y^2 = 1$.

b) Given an explicit expression for f^{-1}.

$f^{-1}(x,y) = (0,y)$.

c) Give an isotopy from D to B.



$h: D \times I \to \mathbb{R}^2$,
$h((0,y),t) = (t\sqrt{1-y^2}, y)$
$h_0 = (0,y)$, $\text{im}(h_0) = y$.
$h_1 = (\sqrt{1-y^2}, y)$, so $\text{im}(h_1) = \text{im}(f) = B$

(easy to check that each h_t is an embedding).

d) Give an isotopy from B to D.

$$h: B\times I \to \mathbb{R}^2, \quad h((x,y),t) = (x(1-t), y)$$

[Lecture ended here]

Facts about isotopies:

Symmetry: If there exists an isotopy from S to T, then there exists an isotopy from T to S.

"Isotopies can be reversed"

Pf: If $h: X\times I \to \mathbb{R}^n$ is an isotopy from S to T, then $\bar{h}: X\times I \to T$, given by $\bar{h}(x,t) = h(x, 1-t)$ is an isotopy from T to S.

Transitivity: If S, T are isotopic and T, U are isotopic, so are S, U.

Sketch of proof: (details omitted, but similar to the argument above That we can always take $X=S$)

Assume $S, T, U \subset \mathbb{R}^n$. We can find isotopies $f: T\times I \to \mathbb{R}^n$ from S to T, $g: T\times I \to \mathbb{R}^n$ from T to U, such that $f_1 = g_0 =$ the inclusion $T \hookrightarrow \mathbb{R}^n$.

Define an isotopy $h: T\times I \to \mathbb{R}^n$ from S to U by $h(x,t) = \begin{cases} f(x, 2t) & \text{for } t\in[0,\frac{1}{2}] \\ g(x, 2t-1) & \text{for } t\in[0,\frac{1}{2}] \end{cases}$

It can be checked that h is indeed continuous. ■

Example: Consider the thick capital letters

X **Y**

Both are isotopic to the disc $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

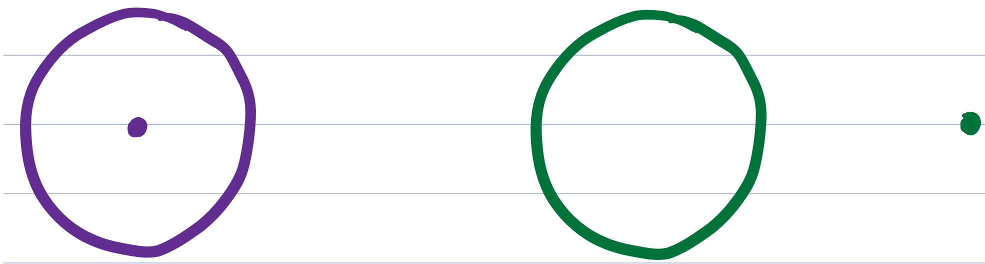
Isotopy from D to **X**

Hence, by transitivity, **X** and **Y** are isotopic.
In particular, they are homeomorphic.

Thus we see that whether two letters are homeomorphic depends on whether we consider the thin or thick versions.

Note: Whether S and T are isotopic depends on where S and T are embedded. (That's not true for homeomorphism!)

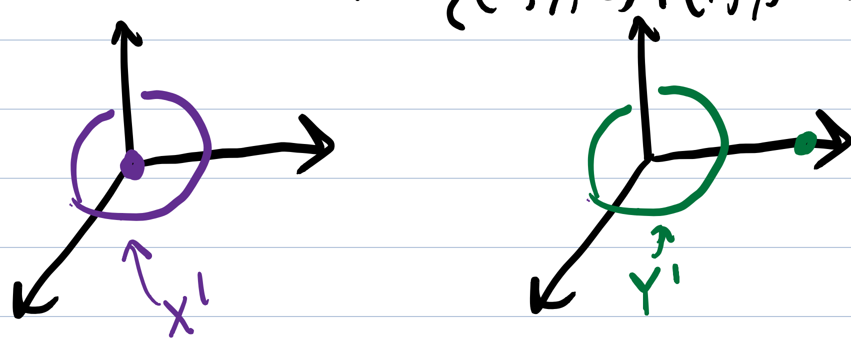
Example $X = S^1 \cup \{0\} \subset \mathbb{R}^2$ $\quad Y = S^1 \cup \{(3,0)\}$.

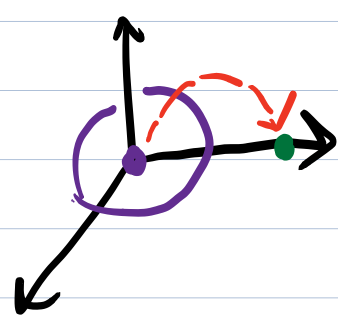


X $\qquad\qquad\qquad\qquad$ Y

X and Y homeomorphic, not isotopic.
But if we embed X, Y in $\mathbb{R}^3$, then they are isotopic there.

That is, let $X' = \{(x,y,0) \mid (x,y) \in X\} \subset \mathbb{R}^3$
$\qquad\qquad\qquad\quad Y' = \{(x,y,0) \mid (x,y) \in Y\} \subset \mathbb{R}^3$





There's an isotopy $h: X' \times I \to \mathbb{R}^3$ which moves the extra point as shown in red.

Similarly, if we embed S and T of the previous example into $\mathbb{R}^4$, they are isotopic there.