AMAT 583 Lec. 8, 9/19/19

Today: Properties of isotopies
Examples of Surprising Isotopies
Equivalence relations ← will be used to define path components.

Isotopy ← here is the main definition from last time

Definition: For $S, T \subset \mathbb{R}^n$ an isotopy from S to T is a homotopy $h: X \times I \to \mathbb{R}^n$ such that

$\mathrm{im}(h_0) = S$, $\mathrm{im}(h_1) = T$,

$h_t: X \to \mathbb{R}^n$ is an embedding for all $t \in I$.

(If there exists an isotopy from S to T, we say S and T are isotopic.)

Facts about isotopies:

Symmetry: If there exists an isotopy from S to T, then there exists an isotopy from T to S.

"Isotopies can be reversed"

Pf: If $h: X \times I \to \mathbb{R}^n$ is an isotopy from S to T, then $\bar{h}: X \times I \to T$, given by $\bar{h}(x,t) = h(x, 1-t)$ is an isotopy from T to S.

Transitivity: If S, T are isotopic and T, U are isotopic, so are S, U.

Sketch of proof: (details omitted, but similar to the argument above that we can always take $X = S$)

Assume $S, T, U \subset \mathbb{R}^n$. We can find isotopies $f : T \times I \to \mathbb{R}^n$ from S to T
$g : T \times I \to \mathbb{R}^n$ from T to U,
such that $f_1 = g_0 =$ the inclusion $T \hookrightarrow \mathbb{R}^n$.

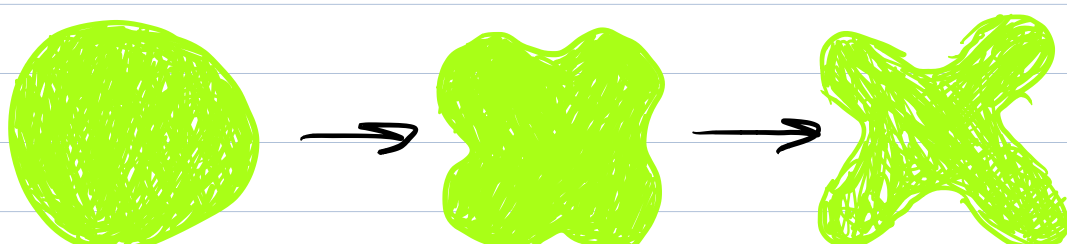
Define an isotopy $h : T \times I \to \mathbb{R}^n$ from S to U by
$$h(x,t) = \begin{cases} f(x, 2t) & \text{for } t \in [0, \tfrac{1}{2}] \\ g(x, 2t-1) & \text{for } t \in [0, \tfrac{1}{2}] \end{cases}$$

It can be checked that h is indeed continuous. ∎

Example: Consider the thick capital letters

X **Y**

Both are isotopic to the disc $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.



Isotopy from D to **X**

Hence, by transitivity, **X** and **Y** are isotopic.
In particular, they are homeomorphic.

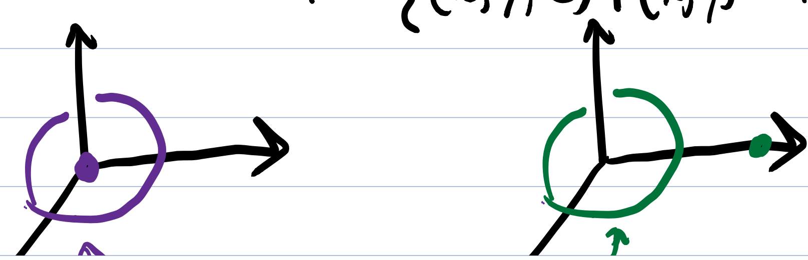
Thus we see that whether two letters are homeomorphic depends on whether we consider the thin or thick versions.

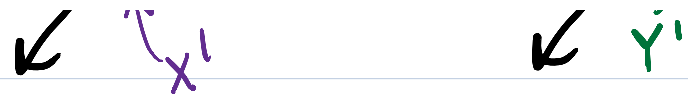
<u>Note</u>: Whether S and T are isotopic depends on where S and T are embedded. (That's not true for homeomorphism!)

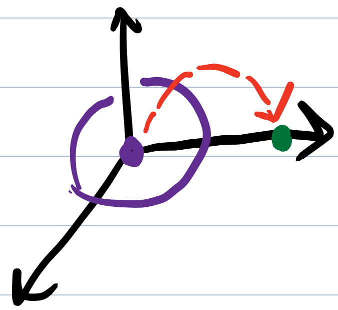
<u>Example</u> $X = S^1 \cup \{0\} \subset \mathbb{R}^2$ $Y = S^1 \cup \{(3,0)\}$.

X and Y homeomorphic, not isotopic.
But if we embed X, Y in $\mathbb{R}^3$, then they are isotopic there.

That is, let $X' = \{(x,y,0) \mid (x,y) \in X\} \subset \mathbb{R}^3$
$Y' = \{(x,y,0) \mid (x,y) \in Y\} \subset \mathbb{R}^3$







There's an isotopy $h: X' \times I \to \mathbb{R}^3$ which moves the extra point as shown in red.

<u>Example</u>: Similarly, returning to the example of non-isotopic objects considered earlier,

These are isotopic when embedded in $\mathbb{R}^4$!

<u>Surprising Isotopies</u>

There are some well-known examples of pairs of objects embedded in $\mathbb{R}^3$ which seem like they ought not to be isotopic, but are. See the posted .pdf slides for some examples of surprising isotopies.

Equivalence relations

You may have not heard this term, but you know many examples of this.

Let S be any set. A relation on S is a function $R: S \times S \to \{0, 1\}$.

"no" "yes"

Notation: Instead of writing $R(x,y) = 1$, we write xRy.
" " " $R(x,y) = 0$, we write $x \not R y$.

slash through the R.

Example: "Less than" $<$ is a relation on $\mathbb{Z}$.
That is, we can think of $<$ as a function
$<: \mathbb{Z} \times \mathbb{Z} \to \{0,1\}$,
e.g. $<(a,b) = 1$ is written as $a < b$
$<(a,b) = 0$ is written as $a \not< b$.

Note: We pretty much never write $<(a,b)$ but the idea that $<$ is a function with domain $\mathbb{Z} \times \mathbb{Z}$ is useful.

awkward!

Example $\leq$, $>$ and $\geq$ are also relations on $\mathbb{Z}$.

Example As in homework 1, let $P(\mathbb{Z})$ denote the power set of $\mathbb{Z}$ = set of all subsets of $\mathbb{Z}$. Then $\subset$ is a relation on $P(\mathbb{Z})$.

In fact, $\subset$ is a relation on $P(S)$ for any set S.

Equivalence relations (often denoted $\sim$)

A relation $\sim$ on S is an **equivalence relation** if

1) $x \sim x \ \forall x \in S$ [reflexivity]
2) $x \sim y$ iff $y \sim x$ [symmetry]
3) $x \sim y,\ y \sim z \Rightarrow x \sim z$ [transitivity]

if $x \sim y$, we say x is equivalent to y.

Example: The equivalence relation $<$ on $\mathbb{Z}$ satisfies only property 3, e.g. $2 \nless 2$, and $3<5$ but $5 \nless 3$.

Example: The relation $\leq$ on $\mathbb{Z}$ satisfies properties 1 and 3, but not 2.

Examples: 1) For any set S, the relation $\sim$ given by $x \sim y \ \forall x, y \in S$ is an equivalence relation.

2) Similarly, the relation $\sim$ given by $x \sim y$ only if $x = y$ is an equivalence relation.