AMAT 583 Lec 9 9/24/19

Today: Path components

Recall from last time:

A relation $\sim$ on a set S is an equivalence relation if

1) $x \sim x \ \forall \ x \in S$ [reflexivity]
2) $x \sim y$ iff $y \sim x$ [symmetry]
3) $x \sim y,\ y \sim z \Rightarrow x \sim z$ [transitivity]

if $x \sim y$, we say x is equivalent to y.

Interesting example: Let $\sim$ be the relation on $\mathbb{Z}$ defined by $a \sim b$ iff $a-b$ is even.

This is an equivalence relation:

Succinct proof:

1) $a-a$ is 0, which is even, $\forall a \in \mathbb{Z}$.
2) $a-b$ is even iff $b-a = -(a-b)$ is even.
3) if $a-b$ is even and $b-c$ is even, then $a-c = (a-b)+(b-c)$ is even, because the sum of two even #'s is even.

Equivalence classes **Def**: For $\sim$ an equivalence relation on S and $x \in S$, let $[x]$ denote the set $\underbrace{\{y \in S \mid y \sim x\}}_{\text{set of all elements of } S \text{ equivalent to } x.} \subset S$.

We call $[x]$ an ***equivalence class*** of $\sim$.

Example: Let $\sim$ be the equivalence relation on $\mathbb{Z}$ given in the previous example.

Q: What is $[0]$? **A**: $z \sim 0$ iff $z-0$ is even iff z is even.

So $[0]$ = the even integers $:= E$

Q: What is $[2]$? **A**: $z \sim 2$ if $z-2$ is even iff z is even.

So $[2] = E$.

In fact, for any even number z, $[z] = E$.

Q: What is $[1]$? **A**: $z \sim 1$ iff $z-1$ is even iff z is odd.

So $[1]$ = the odd integers $:= O$.

Similarly, for any odd z, $[z] = O$.

So there are just two equivalence classes for this relation, E and O.

Fact: For any equivalence relation $\sim$ on a set S, every element of S is contained in exactly one equivalence class of $\sim$.

Pf: For $x \in S$, $x \in [x]$ because $\sim$ is reflexive.

Suppose $x \in [z]$. $[z] = \{y \in S \mid y \sim z\}$. So $x \sim z$.

and thus $z \sim x$. If $y \in [z]$, then $y \sim z$. By transitivity then, $y \sim x$, so $y \in [x]$. This shows that $[z] \subset [x]$. A very similar little argument shows that $[x] \subset [z]$. Thus $[x] = [z]$. This shows that x belongs to exactly one equivalence class, namely $[x]$. ∎

<u>Notation</u>: $S/\sim$ denotes the set of equivalence classes of of $\sim$.

<u>Example</u>: Let $\sim$ be the equivalence relation on $\mathbb{Z}$ of the previous examples.

Then $\mathbb{Z} = \{E, O\}$.

<u>Lemma</u>: For any equivalence relation $\sim$ on a set S and $x, y \in S$, $x \sim y$ iff $[x] = [y]$.

<u>Pf</u>: Assume $x \sim y$. Then by transitivity, $z \sim y \Rightarrow z \sim x$. So $[y] \subset [x]$.
By the same reasoning, if $z \sim x$, then $z \sim y$, so $[x] \subset [y]$.

Since $[x] \subset [y]$ and $[y] \subset [x]$, we have $[x] = [y]$.

Conversely, assume $[x]=[y]$. Then since $x \sim x$, $x \in [x] = [y]$, so $x \sim y$. ∎

Path Components

Recall from homework #2: For a space S and $x, y \in S$, a path from x to y is a continuous function $\gamma: I \to S$ such that $\gamma(0)=x$, $\gamma(1)=y$.

subset of Euclidean space

Define a relation $\sim$ on S by $x \sim y$ iff $\exists$ a path from x to y.

Proposition: $\sim$ is an equivalence relation.

(Proof was not covered in class last time.)

Pf: **Reflexivity:** For $x \in S$, the path $\gamma: I \to S$, given by $\gamma(t)=x$ $\forall t \in I$, is a path from x to itself.

Symmetry: If γ is a path from x to y, then $\bar{\gamma}: I \to S$, $\bar{\gamma}(t) = \gamma(1-t)$ is a path from y to x.

Transitivity: If α is a path from x to y, and β is a path from y to z, then a path γ from x to z is given by

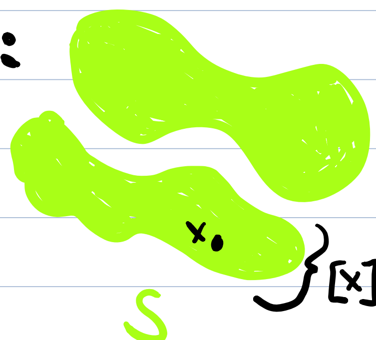
$$\gamma(t) = \begin{cases} \alpha(2t) & \text{for } t \in [0, \tfrac{1}{2}] \end{cases}$$

$\gamma : I \to S, \quad \gamma(t) = \begin{cases} \beta(2t-1) \text{ for } t \in [\frac{1}{2}, 1]. \end{cases}$

Definition: A *path component* of S is an equivalence class of $\sim$, i.e. an element of $S/\sim$.

Illustration:

The set $S \subset \mathbb{R}^2$ shown has two path components.



Notation: $S/\sim$ is written as $\underbrace{\pi(S)}_{\text{The set of path components of } S.}$.

Definition: S is *path connected* if $\pi(S)$ contains exactly one element. Note: If S is non-empty, this is equivalent to the def. of path connected in HW #2.

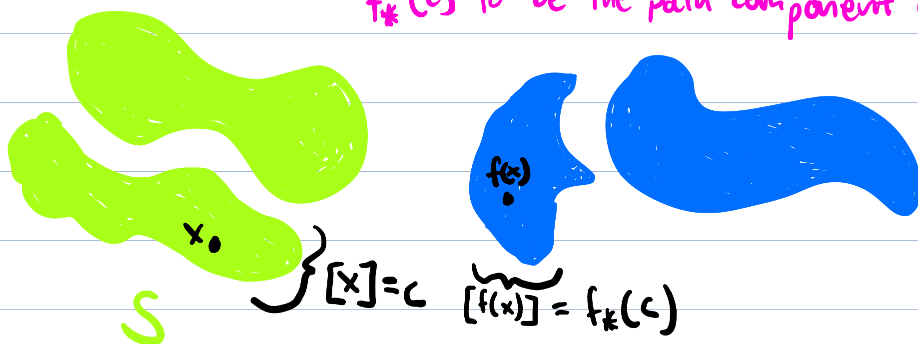
Proposition: If S and T are homeomorphic, then there is a bijection from $\pi(S)$ to $\pi(T)$.

Thus, if S has k path components, so does T.

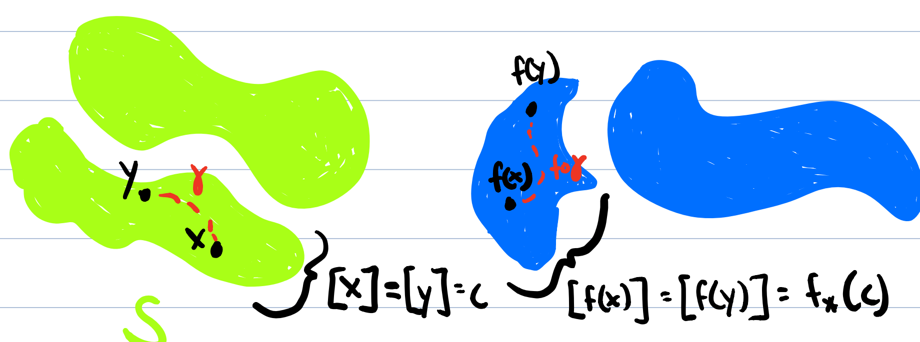
Proof: For any continuous function $f: S \to T$, we define a function: $f_*: \pi(S) \to \pi(T)$ by

$$f_*([x]) = [f(x)]$$

In other words, for c a path component of S, define $f_*(c)$ by choosing $x \in c$ and taking $f_*(c)$ to be the path component of T containing $f(x)$.



Note: We need to check that this definition doesn't depend on the choice of $x \in c$.



That is, we need to check that if $[x]=[y]$ then $[f(x)]=[f(y)]$.

If $[x]=[y]$, then $x \sim y$, i.e., there is a path $\gamma: I \to S$ from x to y. $f \circ \gamma: I \to T$ is a path from $f(x)$ to $f(y)$, so $f(x) \sim f(y)$, which implies $[f(x)]=[f(y)]$. ✓

We'll show that f_* is invertible, hence a bijection, when f is a homeomorphism.

For this, we need two facts:

1) For any $S \subseteq \mathbb{R}^n$, and $Id^S : S \to S$ the identity map, (i.e., $Id^S(x) = x \ \forall x$),

$Id^S_* = Id^{\pi(S)} : \pi(S) \to \pi(S)$.

<u>Pf:</u> $Id^S_*([x]) = [Id(x)] = [x]$.

2) For any continuous maps $f: S \to T$, $g: T \to U$,

$(g \circ f)_* = g_* \circ f_* : \pi(S) \to \pi(U)$

<u>Pf:</u> $(g \circ f)_*([x]) = [g \circ f[x]] = [g(f(x))] = g_*([f(x)]) = g_*(f_*[x]) = g_* \circ f_*([x])$.

Now assume $f: S \to T$ is a homeomorphism.
Then f, f^{-1} are both continuous, and we have
$f^{-1} \circ f = Id^S$
$f \circ f^{-1}\ Id^G$

$$\text{Thus, } (f^{-1} \circ f)_* = \overset{\overset{Id^{\pi(S)}}{\shortparallel}}{Id_*^S} \Rightarrow f_*^{-1} \circ f_* = Id^{\pi(S)}$$

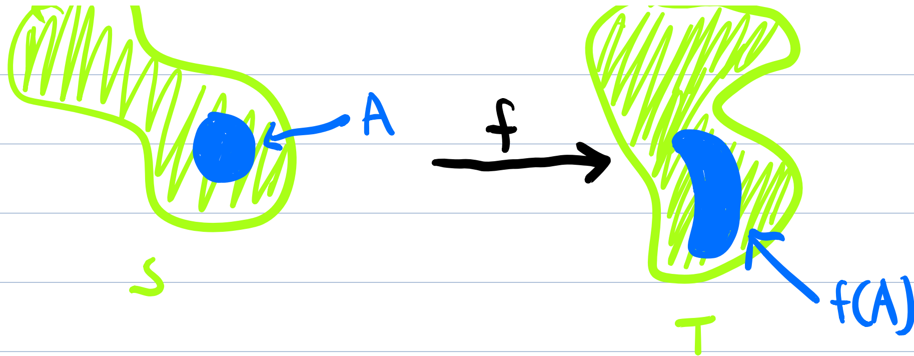
$$(f \circ f^{-1})_* = \underset{\underset{Id^{\pi(T)}}{\shortparallel}}{Id_*^T} \Rightarrow f_* \circ f_*^{-1} = Id^{\pi(T)}.$$

Thus, $f_*: \pi(S) \to \pi(T)$ is invertible, with inverse f_*^{-1}. $\blacksquare$

<u>Application</u>: Consider the symbols $+$, $=$, and $\div$ as subsets of $\mathbb{R}^2$.
$\pi(+) = 1$, $\pi(=) = 2$, $\pi(\div) = 3$. Thus none is homeomorphic to any other.

<u>Application</u>: We prove that as unions of curves w/ no thickness, X and Y are not homeomorphic.

<u>Fact</u>: If $f: S \to T$ is a homeomorphism and $A \subset S$, then A and $f(A)$ are homeomorphic, where $f(A) = \{y \in T \mid y = f(x) \text{ for some } x \in A\}$.



<u>proof of fact:</u> (to be skipped in class) Let $j: A \to S$ be the inclusion. $\text{im}(f\circ j) = f(A)$. Since f is a bijection, so $\widetilde{f\circ j}: A \to f(A)$. It follows from the facts about continuity stated in an earlier lecture that $\widetilde{f\circ j}$ is continuous. Moreover, if $j': f(A) \to T$ is the inclusion, $(\widetilde{f\circ j})^{-1} = \widetilde{f^{-1}\circ j'}$, and this is continuous by the same reasoning.

<u>Proof that X and Y are not homeomorphic:</u>

Let $X' \subset X$ be obtained by removing the center point p. $|\pi(X')| = 4$. Note that there no way to remove a single point from Y to get $Y' \subset Y$ with $|\pi(Y)| = 4$

If we have a homeomorphism $f: X \to Y$, then $f(X')$ is obtained from Y by removing $f(p)$, and $|\pi(f(X'))| = |\pi(X')| = 4$ by the prop., which is impossible. Thus, no homeomorphism $f: X \to Y$ can exist.

[illegible]