AMAT 584 Homework 2 Solutions

Problem 1. 1. For each of the following abstract simplicial complexes, sketch the geometric realization (up to homoeomorphism) and compute the Euler characteristic:

a. $\{[a],[b],[c],[a,b]\}$, **Answer**: E.C.=2

b. $\{[a],[b],[a,b]\}$, **Answer**: E.C.=1

c. $\{[a],[b],[c],[d],[a,b],[c,d]\}$, **Answer**: E.C.= 2

d. $\{[a],[b],[c],[d],[a,b],[b,c],[a,c],[c,d]\}$, **Answer**: E.C.=0.

e. $\{[a],[b],[c],[a,b],[b,c],[a,c],[a,b,c]\}$, **Answer**: E.C.=1.

Which pairs of these simplicial complexes have homotopy equivalent geometric realizations? Which pairs have equal Euler characteristics?
Answer: a. and c. are homotopy equivalent, as both deformation retract onto a pair of points, and both have Euler characteristic 2. Also b. and e. are homotopy equivalent, as both deformation retract onto a single point, and both have Euler characteristic 1.

Problem 2. Let $X = \{(0,0),(2,0),(0,1)\}$.

a. Give an explicit expression for $\check{\text{C}}\text{ech}(X,r)$ for each $r \geq 0$. (Here and forever after, use the closed-ball definition of $\check{\text{C}}\text{ech}(X,r)$.) HINT: To compute the value of r at which the 2-simplex $[(0,0),(2,0),(0,1)]$ first appears in $\check{\text{C}}\text{ech}(V,r)$, it will be helpful to note that $x = (1,.5)$ is the midpoint of the line segment from (0,1) to $(2,0)$, and

$$d(x,(0,0)) = d(x,(2,0)) = d(x,(0,1)) = \frac{\sqrt{5}}{2}.$$

b. Give an explicit expression for $\text{Rips}(X,r)$ for each $r \geq 0$.

c. The set $\text{Vor}(X) = \{\text{Vor}(x) \mid x \in X\}$ is called the *Voronoi decomposition of* X. Sketch $\text{Vor}(X)$. In other words, sketch each of the Voronoi cells of X in a single diagram.

d. Give an explicit expression for $\text{Del}(X,r)$ for each $r \geq 0$.

Let

$$A = (0,0),\ B = (2,0),\ C = (0,1).$$

$\mathrm{Rips}(X,r) = \check{\mathrm{C}}\mathrm{ech}(X,r) = \mathrm{Del}(X,r) =$

$$\begin{cases} \{[A],[B],[C],\} & \text{if } 0 \le r < \frac{1}{2}, \\ \{[A],[B],[C],[A,C]\} & \text{if } \frac{1}{2} \le r < 1, \\ \{[A],[B],[C],[A,B],[A,C]\} & \text{if } 1 \le r < \frac{\sqrt{5}}{2}, \\ \{[A],[B],[C],[A,B],[A,C],[B,C],[A,B,C]\} & \text{if } \frac{\sqrt{5}}{2} \le r. \end{cases}$$

Problem 3. Let $X = \{(0,0),(2,0),(0,2),(2,2)\}$. Give an explicit expression for $\mathrm{Rips}(X,r)$ for each $r \ge 0$. **Answer**: Let

$$A = (0,0),\ B = (2,0),\ C = (0,1).$$

$$\mathrm{Rips}(X,r) = \begin{cases} \{[A],[B],[C],[D]\} & \text{if } 0 \le r < 1, \\ \{[A],[B],[C],[D],[A,B],[B,C],[C,D],[A,D]\} & \text{if } 1 \le r \le \sqrt{2}, \\ \text{The 3-simplex with vertices } A,B,C,D & \text{if } \sqrt{2} \le r. \end{cases}$$

Problem 4. Prove that for any finite $X \subset \mathbb{R}^n$, $\mathrm{Rips}(X,r) \subset \check{\mathrm{C}}\mathrm{ech}(X,2r)$. HINT: Use the triangle inequality.
Answer: Suppose $\sigma = \{x_0, x_k\} \in \mathrm{Rips}(X,r)$. Then for $i \in \{0,k\}$, $\|x_0 - x_i\| \le 2r$, so

$$x_0 \in B(x_0,2r) \cap B(x_1,2r) \cap \cdots \cap B(x_k,2r).$$

Thus the above intersection of balls in non-empty, which implies that $\sigma \in \check{\mathrm{C}}\mathrm{ech}(X,2r)$.

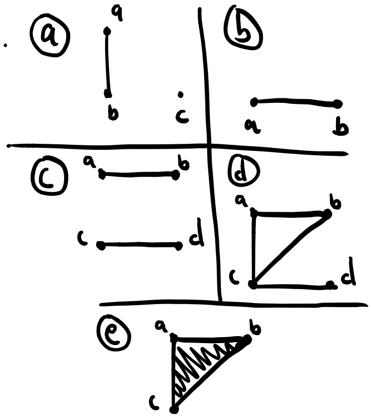
Problem 5. Give an example of a finite set $X \subset \mathbb{R}^2$ and $0 \le r < s$ such that $\mathrm{Rips}(X,r)$ is a connected graph and $\mathrm{Rips}(X,s)$ is a 4-dimensional simplicial complex.

Answer: Here's one possibility among many:

$$X = \{(0,0),(1,0),(2,0),(3,0),(4,0)\},$$

$r = \frac{1}{2}$, $s = 2$.

1.



2c.