AMAT 584 Homework 3 Solutions

Problem 1. Compute the Euler characteristic curves of the Vietoris-Rips filtrations of the following sets $X \in \mathbb{R}^2$:

a. $X = \{(0,0),(2,0),(0,1)\}$,

b. $X = \{(0,0),(2,0),(0,2),(2,2)\}$.

HINT: In homework 2, you gave an explicit expression for the Vietoris-Rips filtration of each of these sets X. To avoid repeating that work, you can assume these as given.

Answer:

a. $\chi(\mathrm{VR}(X)) = \begin{cases} 3 & \text{if } 0 \leq r < \frac{1}{2}, \\ 2 & \text{if } \frac{1}{2} \leq r < 1, \\ 1 & \text{if } 1 \leq r. \end{cases}$

b. $\mathrm{VR}(X,r) = \begin{cases} 4 & \text{if } 0 \leq r < 1, \\ 0 & \text{if } 1 \leq r \leq \sqrt{2}, \\ 1 & \text{if } \sqrt{2} \leq r. \end{cases}$

Problem 2. Show that F_4 is not a field. HINT: Find a non-zero element in F_4 with no multiplicative inverse.
Answer: In F_4, $2 \times 0 = 0$, $2 \times 1 = 2$, $2 \times 2 = 0$, $2 \times 3 = 2$. So 2 has no multiplicative inverse.

Problem 3. Let set S denote the set of all polynomials in one variable with real coefficients. For example,

$$3 + \frac{1}{2}x + 7x^2 + 4x^3 \in S.$$

S has a familiar definition of addition and multiplication. Moreover, S has an additive identity, the constant polynomial 0, and a multiplicative identity, the constant polynomial 1. Is S a field? Explain your answer.
Answer: No. The product of non-zero polynomials of degrees a and b has degree $a + b$. Thus, a polynomial of degree at least 1 has no multiplicative inverse. For example, x has no multiplicative inverse.

Problem 4. Describe all subspaces of the following vector spaces:

a. F_2^2, **Answer**: $\{\vec{0}\}$, $\{\vec{0},(1,0)\}$, $\{\vec{0},(0,1)\}$, $\{\vec{0},(1,1)\}$, F_2^2.

b. F_3^2. **Answer**: $\{\vec{0}\}$, $\{\vec{0}, (1,0), (2,0)\}$, $\{\vec{0}, (0,1), (0,2)\}$, $\{\vec{0}, (1,1), (2,2)\}$, $\{\vec{0}, (2,1), (1,2)\}$, F_3^2.

Problem 5. Let V be a vector space over a field F. Racall that $\vec{0}$ denotes the additive identity of V, and 0 denotes the additive identity of F. Prove the following:

a. For all $a \in F$, $a\vec{0} = \vec{0}$. **Answer**: $a\vec{0} = a(\vec{0}+\vec{0}) = a\vec{0}+a\vec{0}$. Adding $-(a\vec{0})$ to both sides gives $0 = a\vec{0}$.

b. For all $\vec{v} \in V$, $0\vec{v} = \vec{0}$. **Answer**: This is similar to the above. $0\vec{v} = (0+0)\vec{v} = 0\vec{v} + 0\vec{v}$. Adding $-0\vec{v}$ to both sides gives $\vec{0} = 0\vec{v}$.

Note that these proofs only use the axioms for an abstract vector space.