AMAT 584 Homework 4

Due Friday, April 10

Problem 1. Which of the following subsets of $\mathbb{R}^2$ is a subspace? For subsets which are not a subspace, explain your answer.

a. $\{(x, y) \mid x = 3y, y \geq 0\}$,

b. $\{(x, y) \mid x = 3y\}$,

c. $\{(x, 3) \mid x \in \mathbb{R}\}$,

d. $\{(x, x^2) \mid x \in \mathbb{R}\}$.

Problem 2. For each of the following pairs of sets X and Y, compute the symmetric difference of X and Y:

a. $X = \{1, 2, 3\}$, $Y = \{2, 3, 4\}$,

b. $X = \{1, 2, 3\}$, $Y = \{1, 2, 3\}$,

c. $X = \{1, 2, 3\}$, $Y = \{4, 5, 6\}$.

Problem 3. For each of the following subsets S of F_2^4, say whether S is linearly independent, and find a basis for Span(S).

a. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

b. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

HINT: Form a matrix A with the elements of S as rows. (These rows can be in any order.) Do Gaussian elimination on A to form a matrix A'. Standard linear algebra (which you may appeal to here) tells us that

1. S is linearly independent if and only if A' contains no non-zero rows, and
2. the non-zero rows of A' are a basis for Span(S).

Problem 4. Let $T = \{a, b, c\}$. [BONUS] Regard the power set $P(T)$ as a vector space over F_2, as in class. For each of the following subsets $S \subset P(T)$, say whether S is linearly independent, and find a basis for $\mathrm{Span}(S)$.

a. $S = \{\{a, b\}, \{b, c\}, \{a, b, c\}\}$,

b. $S = \{\{a, b\}, \{b, c\}, \{a, c\}\}$.

[HINT: For V any finite dimensional vector space over a field F and B a basis for V, the function $\gamma : V \to F^{|B|}$ defined by $\gamma(v) = [v]_B$ is easily checked to be an isomorphism. If $f : V \to W$ is any isomorphism of vector spaces, f maps linearly independent sets to linearly independent sets. Now recall that we may identity T with a basis for $P(T)$. Represent elements of $P(T)$ as vectors in F_2^3, with respect to this basis, and carry out the computation as in the previous problem.]

Problem 5. Consider the linear map $f : F_2^3 \to F_2^3$ given by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

a. Compute a basis for $\ker f$. [HINT: To do this, you can use the usual Gaussian Elimination + backsolve approach that you learned in your linear algebra class for solving linear systems.]

b. Compute a basis for $\mathrm{im}\, f$. [HINT: $\mathrm{im}\, f$ is the span of the columns of A.]

Problem 6. Repeat the computations of the problem above, but now taking

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Problem 7. Suppose $f : F_2^2 \to F_2^3$ is a linear map such that $f(1, 1) = (1, 1, 0)$ and $f(0, 1) = (0, 1, 0)$. Represent f as a matrix with respect to the standard bases for F_2^2 and F_2^3.

Problem 8. Suppose $g : F_2^3 \to F_2^3$ is a linear map such that

$$\begin{aligned} g(1, 1, 1) &= (1, 0, 0), \\ g(1, 1, 0) &= (0, 1, 0) \\ g(0, 1, 0) &= (0, 0, 1). \end{aligned}$$

Represent g as a matrix with respect to the standard basis for F_2^3.

Problem 9. For f and g as in the previous two problems, represent $g \circ f$ and g as a matrix with the respect to the standard bases for F_2^2 and F_2^3.

Problem 10. Prove that a linear map $f : V \to W$ is an injection if and only if $\ker f = \{\vec{0}\}$.