AMAT 584 Homework 4

Due Friday, April 10

Problem 1. Which of the following subsets of $\mathbb{R}^2$ is a subspace? For subsets which are not a subspace, explain your answer.

a. $\{(x, y) \mid x = 3y, y \geq 0\}$, **Answer**: Not a subspace. Not closed under scalar multiplication.

b. $\{(x, y) \mid x = 3y\}$, **Answer**: It's a subspace. It's easy to check that this is closed under addition and scalar multiplication.

c. $\{(x, 3) \mid x \in \mathbb{R}\}$, **Answer**: Not a subspace. Not closed under scalar multiplication or addition.

d. $\{(x, x^2) \mid x \in \mathbb{R}\}$. **Answer**: Not a subspace. Not closed under scalar multiplication or addition.

Problem 2. For each of the following pairs of sets X and Y, compute the symmetric difference of X and Y:

a. $X = \{1, 2, 3\}$, $Y = \{2, 3, 4\}$, **Answer**: $\{1, 4\}$

b. $X = \{1, 2, 3\}$, $Y = \{1, 2, 3\}$, **Answer**: $\{\}$.

c. $X = \{1, 2, 3\}$, $Y = \{4, 5, 6\}$. **Answer**: $\{1, 2, 3, 4, 5, 6\}$.

Problem 3. For each of the following subsets S of F_2^4, say whether S is linearly independent, and find a basis for Span(S).

a. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

b. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

HINT: Form a matrix A with the elements of S as rows. (These rows can be in any order.) Do Gaussian elimination on A to form a matrix A'. Standard linear algebra (which you may appeal to here) tells us that

1. S is linearly independent if and only if A' contains no non-zero rows, and
2. the non-zero rows of A' are a basis for $\mathrm{Span}(S)$.

Answer: a. Yes, linearly independent. S is thus a basis for $\mathrm{Span}(S)$.
b. Not linearly independent: The third column is a sum of the first two. A basis computed as in the hint is

$$S' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Problem 4 (BONUS)**.** Let $T = \{a, b, c\}$. Regard the power set $P(T)$ as a vector space over F_2, as in class. For each of the following subsets $S \subset P(T)$, say whether S is linearly independent, and find a basis for $\mathrm{Span}(S)$.

a. $S = \{\{a, b\}, \{b, c\}, \{a, b, c\}\}$,

b. $S = \{\{a, b\}, \{b, c\}, \{a, c\}\}$.

[HINT: For V any finite dimensional vector space over a field F and B a basis for V, the function $\gamma : V \to F^{|B|}$ defined by $\gamma(v) = [v]_B$ is easily checked to be an isomorphism. If $f : V \to W$ is any isomorphism of vector spaces, f maps linearly independent sets to linearly independent sets. Now recall that we may identity T with a basis for $P(T)$. Represent elements of $P(T)$ as vectors in F_2^3, with respect to this basis, and carry out the computation as in the previous problem.]

Answer: a. Linearly independent, so S is a basis for $\mathrm{Span}(S)$.
b. Not linearly independent, as the third element is the sum of the first two. $\{\{a, b\}, \{b, c\}\}$ is a basis for $\mathrm{Span}(S)$.

Problem 5. Consider the linear map $f : F_2^3 \to F_2^3$ given by

$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

a. Compute a basis for $\ker f$. [HINT: To do this, you can use the usual Gaussian Elimination + backsolve approach that you learned in your linear algebra class for solving linear systems.]

b. Compute a basis for $\mathrm{im}\, f$. [HINT: $\mathrm{im}\, f$ is the span of the columns of A.]

Answer: a. Applying Gaussian elimination to A gives the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Applying backsolve gives that the solution set to $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ is

$$\left\{ z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| z \in F_2 \right\}.$$

This is exactly the set of elements of $\ker(f)$. Thus,

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| z \in F_2 \right\}$$

is a basis for $\ker(f)$.

b. The following is a basis for $\operatorname{im} f$, obtained using the method of problem 3:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Problem 6. Repeat the computations of the problem above, but now taking

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Answer: a. Solving the linear system as above, we get that as a set,

$$\ker(f) = \left\{ \begin{pmatrix} z \\ y \\ z \end{pmatrix} \middle| z, y \in F_2 \right\} = \left\{ z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| z, y \in F_2 \right\}.$$

Thus

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for $\ker(f)$.

Problem 7. Suppose $f : F_2^2 \to F_2^3$ is a linear map such that $f(1,1) = (1,1,0)$ and $f(0,1) = (0,1,0)$. Represent f as a matrix with respect to the standard

bases for F_2^2 and F_2^3.
Answer:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 8. Suppose $g : F_2^3 \to F_2^3$ is a linear map such that

$$\begin{aligned} g(1,1,1) &= (1,0,0), \\ g(1,1,0) &= (0,1,0) \\ g(0,1,0) &= (0,0,1). \end{aligned}$$

Represent g as a matrix with respect to the standard basis for F_2^3.
Answer:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Problem 9. For f and g as in the previous two problems, represent $g \circ f$ and g as a matrix with the respect to the standard bases for F_2^2 and F_2^3. **Answer**: Multiply the matrices in the previous two problems (see Lecture 27, page 2). We get

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Problem 10. Prove that a linear map $f : V \to W$ is an injection if and only if $\ker f = \{\vec{0}\}$.

Answer: First, we prove that $g(\vec{0}) = \vec{0}$ for any linear map g: We have $g(\vec{0}) = g(\vec{0} + \vec{0}) = g(\vec{0}) + g(\vec{0})$. Adding $-g(\vec{0})$ to both sides gives $g(\vec{0}) = \vec{0}$.

Now suppose $f : V \to W$ is an injection. By the above, $f(\vec{0}) = \vec{0}$, so since f is an injection, we have $f(\vec{v}) \neq 0$ for all $v \neq \vec{0}$. Thus $\ker f = \{\vec{0}\}$. Conversely, suppose $\ker f = \{\vec{0}\}$. Suppose we have $\vec{v}, \vec{v}' \in V$ with $f(v) = f(v')$. Then $\vec{0} = f(v) - f(v') = f(v - v')$, so $v - v' \in \ker f$. But $\ker f = \{0\}$, so $v - v' = 0$. Adding v' to both sides gives $v = v'$. Hence f is an injection.