AMAT 584 Homework 5

Due Wednesday April 15

Problem 1. For each of the following simplicial complexes X

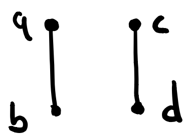
a $X = \{[a], [b], [c], [d], [a, b], [c, d]\}$,

b. $X = \{[a], [b], [c], [d], [e], [a, b], [b, c], [c, d], [a, d], [a, c], [a, e], [b, e], [a, b, c]\}$,

do the following:

1. Sketch the simplicial complex.
2. Represent each non-zero boundary map ∂_j in the chain complex of X as a matrix with respect to the standard bases for $C_j(X)$ and $C_{j-1}(X)$. Use the given order on j-simplices.
3. Compute the dimension of each $Z_j(X)$, $B_j(X)$, and $H_j(X)$, for $j \geq 0$.

Answer: a. The only non-zero boundary matrix is:



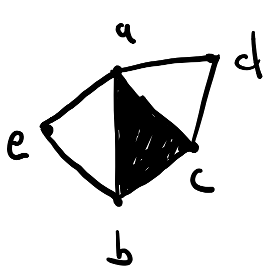
$$[\delta_1] = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$\dim(Z_j)(X)) = \begin{cases} 4 & \text{if } j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim(B_j)(X)) = \begin{cases} 2 & \text{if } j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim(H_j)(X)) = \begin{cases} 2 & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}.$$

b. The non-zero boundary matrices are:



$$[\delta_1] = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \qquad [\delta_2] = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\dim(Z_j)(X)) = \begin{cases} 5 & \text{if } j = 0, \\ 3 & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim(B_j)(X)) = \begin{cases} 4 & \text{if } j = 0, \\ 1 & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim(H_j)(X)) = \begin{cases} 1 & \text{if } j = 0, \\ 2 & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}.$$

Problem 2. For X as in problem 1a., explicitly write down all of the cosets of $H_0(X)$. Sketch each element of each coset.

Answer: There are four cosets: see below for the sketches

$$\begin{aligned} B_0(X) &= \{0,\ [a]+[b],\ [c]+[d],\ [a]+[b]+[c]+[d]\}, \\ [a]+B_0(X) &= \{[a],\ [b],\ [a]+[c]+[d],\ [b]+[c]+[d]\}, \\ [c]+B_0(X) &= \{[c],\ [a]+[b]+[c],\ [d],\ [a]+[b]+[d]\}, \\ ([a]+[c])+B_0(X) &= \{[a]+[c],\ [b]+[c],\ [a]+[d],\ [b]+[d]\}. \end{aligned}$$

Problem 3. For X as in problem 1b., explicitly write down the coset of $H_1(X)$ containing each of the following elements of $Z_1(X)$, and sketch each element of the coset.

a. $z_1 = [a,b]+[b,c]+[a,c]$,

b. $z_2 = [c,d]+[a,d]+[a,c]$,

c. $z_3 = [a,b]+[a,e]+[b,e]$.

Answer: Note that

$$B_1(X) = \{0,\ [a,b]+[b,c]+[a,c]\}.$$

Hence the cosets are

$$\begin{aligned} z_1 + B_1(X) &= B_1(X), \\ z_2 + B_1(X) &= \{[c,d]+[a,d]+[a,c],\ [a,b]+[b,c] \qquad +[c,d]+[a,d]\}, \\ z_3 + B_1(X) &= \{[a,b]+[a,e]+[b,e],\ [a,e]+[b,e]+[b,c]+[a,c]\}, \end{aligned}$$

see below for the sketches

Problem 4. For both simplicial complexes considered in problem 1, give a basis for each non-zero $H_j(X)$. [Hint: $\{z_1, z_2, z_3\}$, as defied in the previous problem, is a basis for $Z_1(X)$.]

Answer:

a. Note that has X has two components, call them C_1 and C_2, with $a \in C_1$ and $b \in C_2$. Thus, according to a proposition from class/the notes,

$$\{[a] + B_0(X),\ [b] + B_0(X)\}$$

is is a basis for $H_0(X)$.

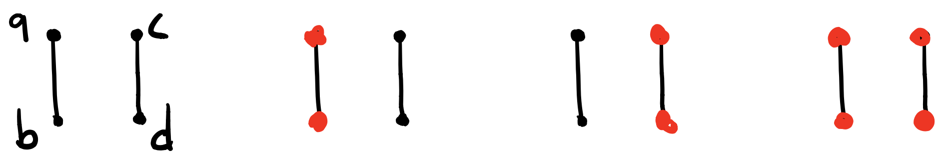
b. The singleton set $\{[a]+B_0(X)\}$ is a basis for $H_0(X)$. Using the answer to the last problem, the hint, and the proposition about bases for homology from lecture, we have that

$$\{z_2 + B_1(X),\ z_3 + B_1(X)\}$$

is a basis for $H_1(X)$.

Sketches for problem 2:

$B_0(X)$



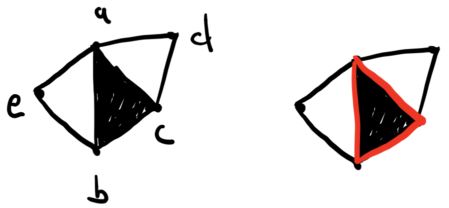
$[a]+B_0(X)$

$[c]+B_0(X)$

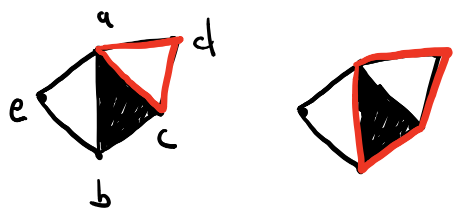
$[a]+[c]+B_0(X)$

Sketches for problem 3:

$Z_1 + B_1(X) = B_1(X)$



$Z_2 + B_1(X)$



$Z_3 + B_1(X)$