AMAT 584 Lecture 10

Today: Čech complexes, Rips complexes.

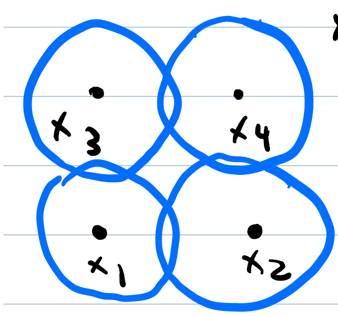
Definition:
For $X=\{x_1,\dots,x_n\}\subset\mathbb{R}^n$ and $r>0$, the Čech complex $\check{C}ech(X,r)$ is the abstract simplicial complex with vertex set $\{x_1,\dots,x_n\}$, such that

$$[x_{j_0},\dots,x_{j_k}]\in \check{C}ech(X,r) \text{ iff}$$

$$B(x_{j_0},r)\cap B(x_{j_1},r)\cap\cdots\cap B(x_{j_k},r)\neq\emptyset.$$

Example from last lecture:

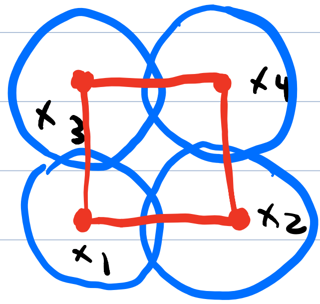
$$\text{Let } X=\{\underset{x_1}{\underset{\parallel}{(0,0)}},\ \underset{x_2}{\underset{\parallel}{(1,0)}},\ \underset{x_3}{\underset{\parallel}{(0,1)}},\ \underset{x_4}{\underset{\parallel}{(1,1)}}\}.$$



Let $r=\frac{1}{2}+\delta$, δ small

I've drawn $B(x_i,r)$ for each $i\in\{1,\dots,4\}$, in blue.

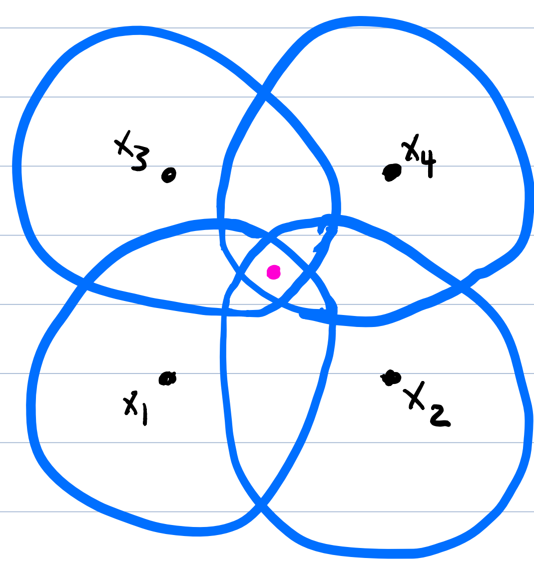
$\check{C}ech(X,r) = \{[x_1],[x_2],[x_3],[x_4],[x_1,x_2],[x_1,x_3],[x_2,x_4],[x_3,x_4]\}$.



$\longleftarrow |\check{C}ech(X,r)|$ is shown red.

Note that $|\check{C}ech(X,r)|$ is homotopy equivalent to $U(X,r)$: $U(X,r)$ deformation retracts onto $|\check{C}ech(X,r)|$.

Now let r' be large enough so that the center point $(\frac{1}{2}\ \frac{1}{2})$ is contained in each closed ball, i.e. $r' \geq \frac{\sqrt{2}}{2}$.



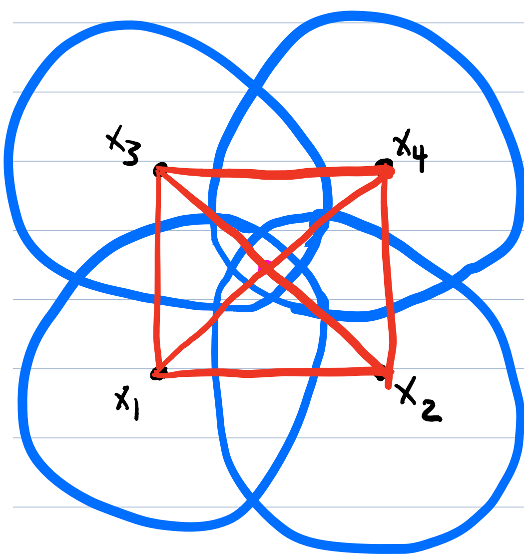
$\check{C}ech(X, r')$ contains all possible simplices on the vertex set $\{x_1, \ldots, x_4\}$, i.e.,

$\check{C}ech(X, r') =$

$\{[x_1], [x_2], [x_3], [x_4], [x_1, x_2], [x_1, x_3], [x_2, x_4], [x_3, x_4], [x_2, x_3], [x_1, x_4],$
$[x_1, x_2, x_3], [x_1, x_2, x_4], [x_1, x_3, x_4], [x_2, x_3, x_4], [x_1, x_2, x_3, x_4]\}$

<u>Definition</u>: The k-skeleton of a simplicial complex is the subcomplex consisting of all simplices of dimension at most k.

Thus, the 1-skeleton of $\check{C}ech(X, r')$ looks like this.



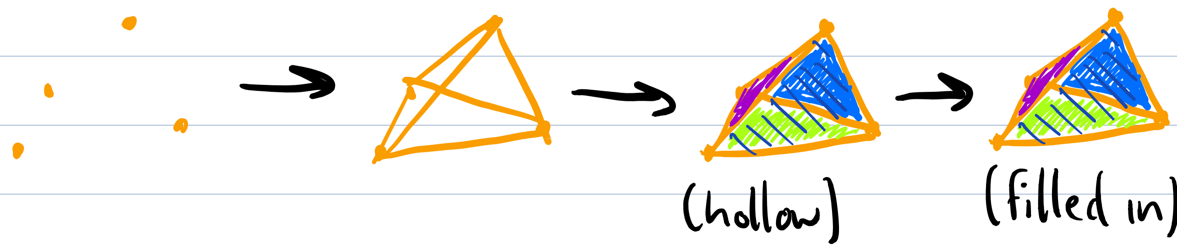
The 1-skeleton is the complete graph on 4 vertices, i.e., every pair of vertices is connected by an edge.

$|\check{C}ech(X, r')| = |Geo(\check{C}ech(X, r'))|$ is the 3-simplex

$[e_1, e_2, e_3, e_4]$ in $\mathbb{R}^4$.

This is a tetrahedron.

A more intuitive way to think about $|\check{C}ech(X, r')|$ is this:
- Start with 4 points.
- Glue in an edge between each pair of points
- For each triple of points, glue in a triangle.
- Glue in the solid tetrahedron.



Note in particular that $|\check{C}ech(X, r')|$ does not sit inside $\mathbb{R}^2$, even though X sits inside $\mathbb{R}^2$.

<u>Remark</u>: In general, if X is an (abstract) simplicial complex with $|V(X)| = k+1$, and X contains all non-empty subsets of $V(X)$, then $|X|$ is a k-simplex.

Thus, in this case, we call X a k-simplex.
Thus, in the context of simplicial complexes, "simplex" has several meanings, but they are all very closely related.

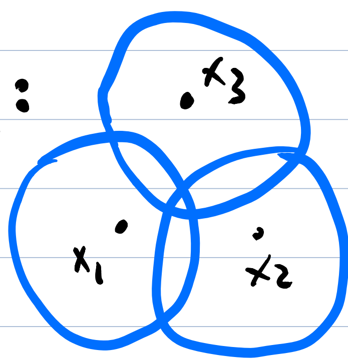
Note that, as above $|\check{C}ech(X,r')|$ and $U(X,r')$ are homotopy equivalent: Both are h.e. to a point.

__Theorem:__ For any finite $X \subset \mathbb{R}^n$ and $r \geq 0$,
$U(X,r) \simeq \check{C}ech(X,r)$.

means "is homotopy equivalent to"

The theorem is not that easy to prove. It is an immediate consequence of a more general result called the __nerve theorem__, which is very important in topology.

__Exercise__:



Consider $X = \{x_1, \ldots, x_3\}$ as above, and r so that the balls $B(x_i, r)$ intersect as shown.

What is $\check{C}ech(X,r)$?
What is $|\check{C}ech(X,r)|$?