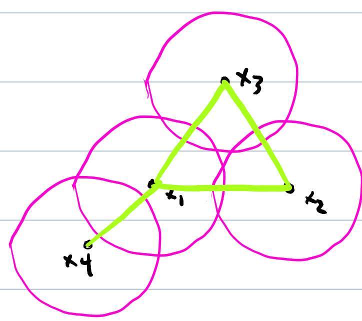
AMAT 584 Lec. 11 Friday Feb. 14 ♥

Today: Vietoris Rips and Alpha Complexes

In the last two lectures, we've been studying Čech complexes of points in $\mathbb{R}^n$.

Example: $X = \{\underbrace{\overset{x_1}{(0,0)}, \overset{x_2}{(2,0)}, \overset{x_3}{(1,\sqrt{3})}}_{\text{vertices of an equilateral triangle}}, \overset{x_4}{(-1,-1)}\}$



Choose $r = 1+\delta$, $\delta \geq 0$ very small.
Note that $B(x_1,r) \cap B(x_2,r) \cap B(x_3,r) = \emptyset$.

Then $\check{C}ech(X,r) =$
$\{[x_1],[x_2],[x_3],[x_4],[x_1,x_2],[x_1,x_3],[x_2,x_3],$
$[x_1,x_4]\}$ (shown in green.)

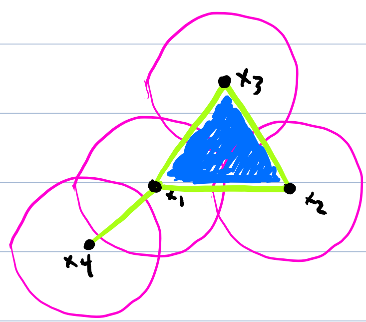
Theorem: For any finite $X \subset \mathbb{R}^n$ and $r \geq 0$,
$\check{C}ech(X,r) \simeq U(X,r)$.

Two limitations of Čech complexes:
1) Computationally expensive
2) Extension to arbitrary finite metric spaces is possible, but awkward

The Vietoris-Rips Complex is an alternative to the Čech complex which is defined for arbitrary finite metric metric space.

It is usually much easier to compute than the Čech complex, though it's computation can also be expensive for larger point clouds.

To a preview of Vietoris-Rips complexes, I will note that in the example considered above, the Vietoris-Rips complex is the same as the Čech complex, but with the triangle filled in.



Clique complexes

Given a graph G (i.e., 1-D simplicial complex), let $CL(G)$ denote the largest abstract simplicial complex with the same 1-skeleton.

Thus, to construct $CL(G)$, start with G and

- For each triple of vertices $x_1, x_2, x_3 \in V(G)$ such that $[x_i, x_j] \in G$ for all $i<j \in \{1,2,3\}$, add the 2-simplex $[x_1, x_2, x_3]$ to $CL(G)$ } Add every possible 2-simplex into $CL(G)$
- For each quadruple of vertices $x_1, x_2, x_3, x_4 \in V(G)$ s.t. $[x_i, x_j] \in G$ for all $i<j \in \{1,2,3,4\}$, add the 3-simplex $[x_1, x_2, x_3, x_4]$ to $CL(G)$ } Add every possible 3-simplex into $CL(G)$

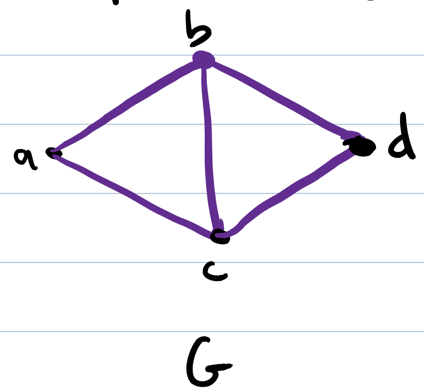
$\vdots$

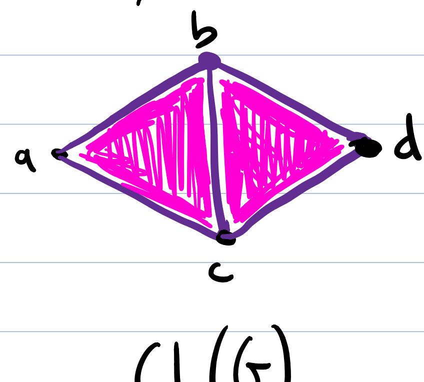
- And so on for higher simplices.

Formal Def:

$$CL(G) = \{[x_1, \ldots, x_k] \in V(G) \mid [x_i, x_j] \in G \text{ for all } i<j \in \{1, \ldots, k\}\}.$$

<u>Example</u>: $G = \{[a],[b],[c],[d],[a,b],[a,c],[b,c],[b,d],[c,d]\}$





Neighborhood Graphs (review)

For X a finite metric space and $r \geq 0$, let $N_r(X)$, the **r-neighborhood graph of X**, be the graph s.t.

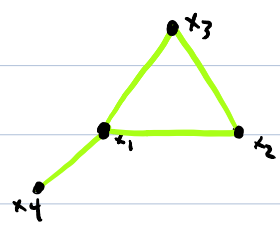
- $V(N_r(X)) = X$ (i.e., the vertex set is X).
- $[y,z] \in N_r(X)$ if and only if $d_X(y,z) \leq r$.

metric on X.

Example: For $X = \{\underbrace{\overset{x_1}{(0,0)}, \overset{x_2}{(2,0)}, \overset{x_3}{(1,\sqrt{3})}, \overset{x_4}{(-1,-1)}}_{\text{example from the start of lecture}}\}$ with the Euclidean metric.

and $r = 2$,

$$N_r(X) = N_2(X) = \{[x_1],[x_2],[x_3],[x_4],[x_1,x_2],[x_1,x_3],[x_2,x_3],[x_1,x_4]\}$$
$$= \text{Cech}(X,1).$$

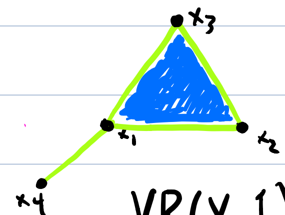


Definition: For X a finite metric space and $r \geq 0$, the Vietoris-Rips complex of X with scale parameter r is $VR(X,r) = CL(N_{2r}(X))$.

In words, $VR(X,r)$ is the clique complex of the $2r$-neighborhood graph of X.

Example: For X as in the last example,

$$VR(X)=\{[x_1],[x_2],[x_3],[x_4],[x_1,x_2],[x_1,x_3],[x_2,x_3],[x_1,x_4],$$
$$[x_1,x_2,x_3]\}$$



$VR(X,1)$

Easy fact: If $X\subset\mathbb{R}^n$, then $\forall r\geq 0$, $N_{2r}(X)$ is the 1-skeleton of $\check{C}ech(X,r)$. Hence,

$VR(X,r)$ is the clique complex of the 1-skeleton of $\check{C}ech(X,r)$

In particular, $\check{C}ech(X,r)\subset VR(X,r)$.

Conversely, it can be shown that $VR(X,r)\subset\check{C}ech(X,\sqrt{2}r)$.

This is non-trivial, but it is easy to show the weaker result that $VR(X,r)\subset\check{C}ech(X,2r)$