AMAT 584 Lecture 12

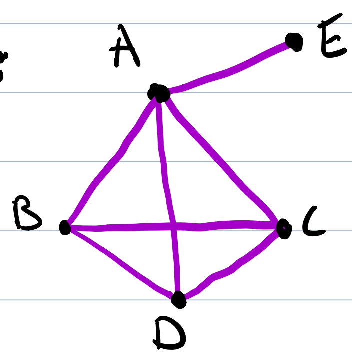
Today: Comparison of Vietoris-Rips vs. Čech complexes.
Alpha complexes.

Review:

For G a graph, a $(k\text{-})$clique of G is a non-empty subset $\{x_1, \ldots, x_k\}$ of $V(G)$, the vertex set of G, such that $[x_i, x_j] \in G$ for all $i < j \in \{1, \ldots, k\}$.

Note: If σ is a clique and $\tau < \sigma$ is non-empty, then τ is a clique.

Example:



$\{A,B,C,D\}$ is a clique
$\Rightarrow \{A,B,C\}$ is a clique.

Hence, the set of all cliques of a graph G is a simplicial complex, which we denote $CL(G)$ and call the <u>clique complex</u> of G.

Def: For X a finite metric space and $r \geq 0$, the Vietoris-Rips complex of r is

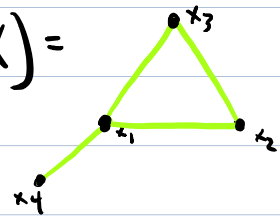
$$VR(X,r) = CL(N_{2r}(X)),$$

where $N_r(X)$ is the r-neighborhood graph of X.

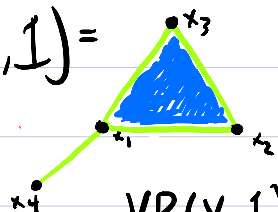
Example: For $X = \{\overset{x_1}{\overset{\shortparallel}{(0,0)}}, \overset{x_2}{\overset{\shortparallel}{(2,0)}}, \overset{x_3}{\overset{\shortparallel}{(1,\sqrt{3})}}, \overset{x_4}{\overset{\shortparallel}{(-1,-1)}}\}$ } from last time

with the Euclidean metric.

$N_2(X) =$



$VR(X,1) =$



$VR(X,1)$

Comparison of Čech and VR-Complexes

Note: from now on, all Čech complexes will be defined in terms of intersections of closed balls, not open balls!

Easy fact: If $X \subset \mathbb{R}^n$, then $\forall r \geq 0$, $VR(X,r)$ and $\check{C}ech(X,r)$ have the same 1-skeleton, namely $N_{2r}(X)$.

Since $VR(X,r) = CL(N_{2r}(X))$, it's the largest simplicial complex with 1-skeleton, which implies

$$\check{C}ech(X,r) \subset VR(X,r).$$

Conversely, it can be shown that

$$VR(X,r) \subset \check{C}ech(X, \sqrt{2}r).$$

This is non-trivial, but it is easy to show the weaker result that $VR(X,r) \subset \check{C}ech(X, 2r)$ using the triangle inequality.

To summarize, we have that if $X \subset \mathbb{R}^n$ and $r \geq 0$, then

$$\check{C}ech(X,r) \subset VR(X,r) \subset \check{C}ech(X, \sqrt{2}r).$$

This has important consequences for persistent homology, though we are not yet ready to discuss them.

Delaunay Complexes (a.k.a. Alpha Complexes)

Given $X \subset \mathbb{R}^n$ and $r > 0$, we will define a subcomplex $Del(X,r) \subset \check{C}ech(X,r)$ which is usually much smaller, and such that the inclusion

$$j : Del(X,r) \hookrightarrow \check{C}ech(X,r)$$

I think that this is one of the most charming constructions in TDA!

is a homotopy equivence.

The construction of $Del(X,r)$ relies on a fudemental concept in computational geometry, Voronoi cells.

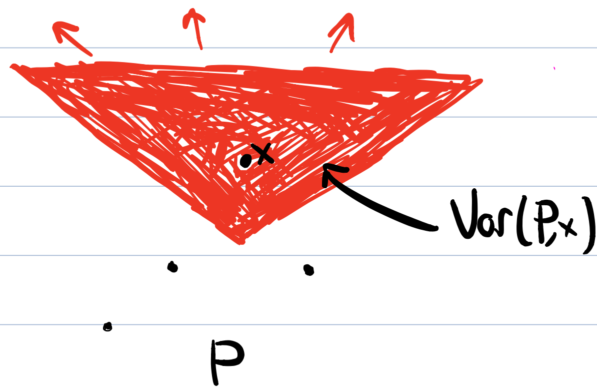
Def: For $P \subset \mathbb{R}^n$ finite and $x \in P$, the Voronoi cell of x is the set

$$Vor(x) = \{ y \in \mathbb{R}^n \mid \|y-x\| \leq \|y-x'\| \text{ for all } x' \in P \}.$$

In words, $Vor(x)$ is the set of points in $\mathbb{R}^n$ which are as close to x as to any other point of P.

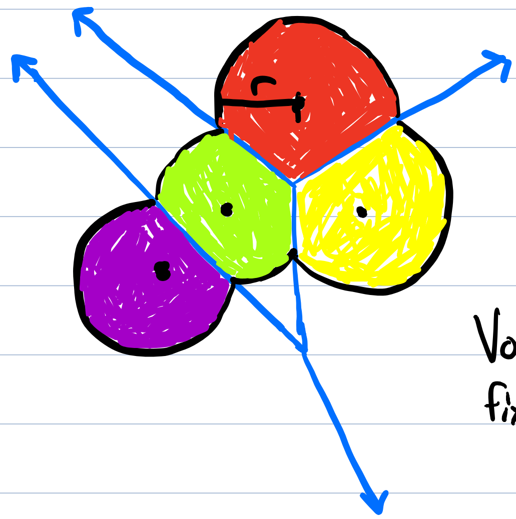
Note that in my notation, the dependence of $Vor(X)$ on P is implicit.

Example: For P and x as shown here,



In general, the Voronoi cells of a finite set $P \subset \mathbb{R}^n$ decompose $\mathbb{R}^n$ into a collection of convex polyhedra, intersecting only along their boundaries

Now, given $P \subset \mathbb{R}^n$, $x \in P$, $r > 0$, let $Vor(x,r) = Vor(x) \cap \underbrace{B(x,r)}_{\text{closed ball of radius } r \text{ centered at } x.}$



$Vor(x,r)$ for each $x \in P$, for some fixed r

Definition: For $X \subset \mathbb{R}^n$ finite and $r \geq 0$, $Del(X,r)$ is the abstract simplicial complex with vertex set X, and containing a simplex $[x_0, \ldots, x_k]$ iff

$$Vor(x_0,r) \cap Vor(x_1,r) \cap \cdots \cap Vor(x_k,r) \neq \emptyset.$$

Example: