Today: Nerves
Filtered simplicial complexes

The Delaunay complexes and Čech complexes are both examples of a general construction of an abstract simplicial complex called a nerve.

Definition: Let $S=\{S_1,\ldots,S_n\}$ be a set of sets.
$\text{Nerve}(S)=\{\{S_{j_0},\ldots,S_{j_k}\}\subset S \mid S_{j_0}\cap S_{j_1}\cap\cdots\cap S_{k}\neq\emptyset\}$.

$\text{Nerve}(S)$ is an abstract simplicial complex.

Example: For $X\subset\mathbb{R}^n$ finite and $r\geq 0$, let
$S=\{B(y,r)\mid y\in X\}$.

Then $\check{C}\text{ech}(X,r)=\text{Nerve}(S)$

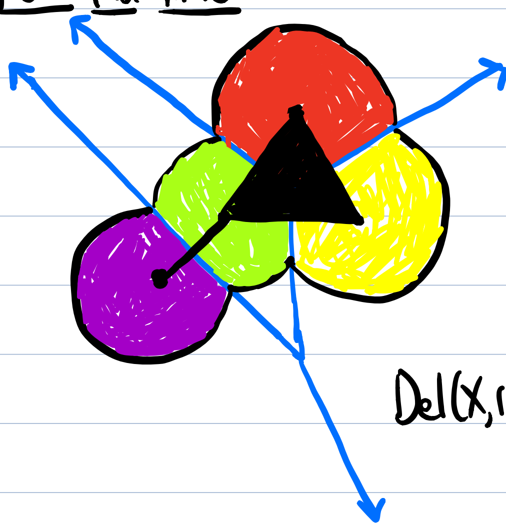
Let $T=\{B(y,r)\cap \text{Vor}(y)\mid y\in X\}$,

where

$$\text{Vor}(y)=\{z\in\mathbb{R}^n \mid \|z-y\|\leq\|z-y'\| \text{ for all } y'\in X\}.$$

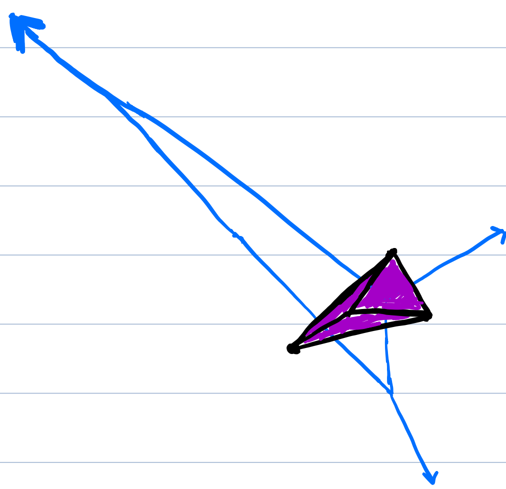
$\text{Del}(X,r)=\text{Nerve}(T)$.

<u>Example from last time:</u> $X = \{(0,0), (2,0), (1,\sqrt{3}), (-1,-1)\}$.



$\mathrm{Del}(X,r)$ (for some r).

<u>Def</u>: For $X \subset \mathbb{R}^n$ finite, let $A = \{\mathrm{Vor}(y) \mid y \in X\}$. $\mathrm{Nerve}(A)$ is called the Delaunay triangulation of X, and is denoted $\mathrm{Del}(X)$.



Note that in this example, $|\mathrm{Del}(X)|$ embeds in the lane.
This is not always the case: e.g., consider the vertices of a square.

However, for $X \subset \mathbb{R}^n$, if no subset of X of size $n+1$ lies on an $(n-1)$-dimensional sphere, then $|\mathrm{Del}(X)|$ embeds in $\mathbb{R}^2$, in the same way as in the example above.

A similar statement holds in $\mathbb{R}^n$.

Clearly, $\mathrm{Del}(X,r) \subset \mathrm{Del}(X)$.

<u>Fact</u>: For r sufficiently large, $\mathrm{Del}(X,r) = \mathrm{Del}(X)$.

To compute $\mathrm{Del}(X,r)$, one usually first computes $\mathrm{Del}(X)$. Computing $\mathrm{Del}(X)$ is a very classical and heavily studied problem in computational geometry.

This is all we will say about computing $\mathrm{Del}(X,r)$, for now. See Edelsbrunner and Harer for more details on computation.

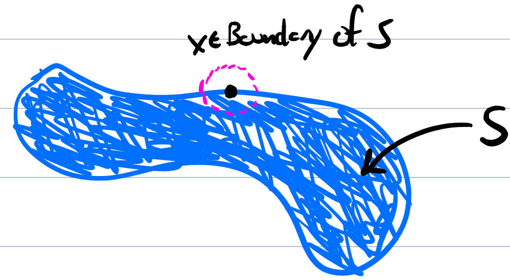
<u>Nerve Theorem</u>: Let $S = \{S_1, \ldots, S_k\}$ be a set of closed convex sets in $\mathbb{R}^n$. Then $N(S) \simeq S_1 \cup S_2 \cup \cdots \cup S_k$.

<u>Corollaries</u>: For $X \subset \mathbb{R}^n$ finite and $r \geq 0$

(i) $\check{C}\mathrm{ech}(X,r) \simeq U(X,r) \simeq \mathrm{Del}(X,r)$

(ii) $\mathrm{Del}(X)$ is contractible.

Closed Sets: For $S \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ is a boundary point of S, if for each $r > 0$, the open ball centered at x of radius r contains at least one point in S and at least on point not in S.



Def: S is said to be closed if it contains all of its boundary points.