AMAT 584 Lecture 15, 2/24/20

Today (and next several lectures): Abstract Linear Algebra

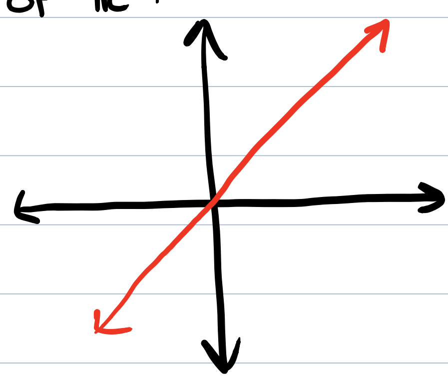
Our Motivation: We will need two big ideas from linear algebra to define and work with homology:

1) The dimension of a vector space,
2) The quotient of a vector space.

Abstract Vector Spaces (Introductory Remarks)

In typical introductory linear algebra classes, one considers the following vector spaces:

- $\mathbb{R}^n$, $n \geq 0$
- $\mathbb{C}^n$, $n \geq 0$
- Subspaces of these, e.g. a line through the origin is a subspace of $\mathbb{R}^2$:



In linear algebra, one is concerned primarily with two operations on vector spaces: addition and scalar multiplication.

For example, $(1,3)+(2,5)=(1+2,3+5)=(3,8)$ (addition)

$7\cdot(1,3)=(7\cdot 1,7\cdot 3)$ (scalar multiplication)

In many places in mathematics, including topology, we need a more abstract definition of a vector space which encompases these examples.

Understanding the general definition well will give you a fuller understanding of the familiar cases of $\mathbb{R}^n$ and $\mathbb{C}^n$!

Fields: The first ingredient for the definition of an abstract vector space is a field.

Definition: A field is a set F, together with functions

$+ : F\times F \to F$ (addition)

$\cdot : F\times F \to F$ (multiplication)

Note: $+(a,b)$ is written as $a+b$

$\cdot(a,b)$ is written as $a\cdot b$.

satisfing all the familiar properties of arithmetic over the rational numbers $\mathbb{Q}$ or real numbers $\mathbb{R}$, namely the following:

Associativity of addition and multiplication:

$$(a+b)+c=a+(b+c)$$
$$(a\cdot b)\cdot c=a\cdot(b\cdot c)$$

Commutativity of addition and multiplication:

$a+b=b+a$
$a\cdot b=b\cdot a$

Additive and multiplicative identities:

There exist distinct elements of F, which we will write as 0 and 1, such that $a+0=a$ and $1\cdot a=a$ $\forall a\in F$.

additive identity

multiplicative identity.

Note: 0 and 1 needn't be the usual integers 0 and 1, but they are when $F=\mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Additive inverses:

$\forall a\in F$, there exists an element in F, denoted $-a$, such that $a+-a=0$

This property implies that subtraction makes sense.

Multiplicative inverses:

$\forall$ non-zero $a\in F$, $\exists$ an element in F denoted a^{-1} or $\frac{1}{a}$, such that that $a\cdot\frac{1}{a}=1$

Distributivity: $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$.

Examples: As suggested above, $\mathbb{Q}$ and $\mathbb{R}$ are examples of fields.

So are the complex numbers $\mathbb{C}$. (Multiplicative inverse of $0 \neq z = a+bi \in \mathbb{C}$ is $\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$

Non-example of a field: The set of integers $\mathbb{Z}$ satisfies all the properties of a field except the existance of multiplicative inverses.

Prime Fields

Let $F_2 = \{0,1\}$. Define $+ : F_2 \times F_2 \to F_2$ and
$\cdot : F_2 \to F_2 \to F_2$

by the following tables

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Then one can check with these choices of addition and multiplication, F_2 is a field.

In practical applications of TDA this is the most important field!

More generally, let p be a prime number, e.g. $p=2, 3, 5$, or 7.

Let $\mathbb{F}_p = \{0, 1, \dots, p-1\}$.

Define $+ : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ by taking $a+b$ to be the remainder of the usual integer sum after dividing by p.

e.g. in $\mathbb{F}_5$, $4+4=3$.

Similarly, define $\cdot : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ by taking $a \cdot b$ to be the remainder of the usual integer product after dividing by p.

e.g. in $\mathbb{F}_5$, $4 \cdot 4 = 1$.

With these choices of addition and multiplication, $\mathbb{F}_p$ is a field.