AMAT 584 Lecture 17, 2/28/20

Today: Finish with subspaces
Dimension of a vector space + Bases (start)

Review

Definition: A vector space over a field F is a set V together with functions

$+ : V \times V \to V$ (addition)
$\cdot : F \times V \to V$ (scalar multiplication)

Note: $+(a,b)$ is written as $a+b$
$\cdot(a,b)$ is written as $a \cdot b$ or ab.

Satisfying all the usual properties of addition and scalar multiplication of vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$.

Here's an important example not mentioned last time:

For any field F and n a positive integer,

$F^n = \{(x_1, \ldots, x_n) \mid x_i \in F\}$ is a vector space with addition and scalar multiplication given by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$
$$c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n).$$

Example: F_2^2 is the vector space with underlying set $\{(0,0),(1,0),(0,1),(1,1)\}$.

additive identity. We write this as $\vec{0}$.

Subspaces, continued

Review

Def: A subspace of a vector space V over F is a subset $W \subset V$ such that

$$\vec{w}_1 + \vec{w}_2 \in W \quad \forall\ \vec{w}_1, \vec{w}_2 \in W$$
$$a\vec{w} \in W \quad \forall\ a \in F,\ \vec{w} \in W.$$

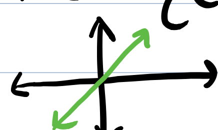
Fact: The restriction of the addition and scalar mult. operations to W gives W the structure of a vector space.

Examples:

- Any vector space V is a subspace of itself.
- For any vector space V, $\{\vec{0}\}$ is a subspace of V.

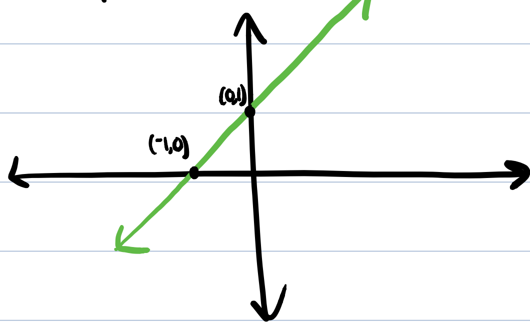
  additive identity of V.

- For any $c \in \mathbb{R}$, the line $\{(x,y) \mid y = cx\}$ is a subspace of $\mathbb{R}^2$.



- $\{(x,y,0) \mid x,y \in \mathbb{R}\}$ = the plane $z=0$ in $\mathbb{R}^3$, is a subspace of $\mathbb{R}^3$.

· $W = \{(x,y) \mid y = x+1\}$ is <u>not</u> a subspace of $\mathbb{R}^2$



e.g. $(-1,0), (0,1) \in W$, but $(-1,0)+(0,1) = (-1,1) \notin W$.

In general, a subspace of a vector space always contains $\vec{0}$!

- The set of all continuous (or differentiable, or polynomial) functions is a subspace of $\underbrace{\mathrm{Fun}(\mathbb{R}, \mathbb{R})}$.

  vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

- $\{(0,0), (1,0)\}$ is a subspace of $\mathbb{F}_2^2$.

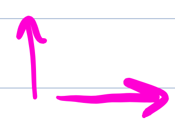
<u>Exercise</u>: Determine all subspaces of $\mathbb{F}_2^2$.

Bases + Dimension

Every abstract vector space has a dimension.
This is either a non-negative integer or ∞.

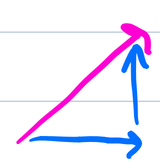
Intuitively, the dimension is the number of independent directions in the vector space.

For example, in $\mathbb{R}^2$, there are two independent directions

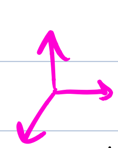


So $\mathbb{R}^2$ has dimension 2.

There are other directions as well, e.g. , but this is not independent of the first two, as I can always move in this direction by first moving right and then moving up.



Similarly, $\mathbb{R}^3$ has 3 independent directions, so has dimension 3,



And $\mathbb{R}$ has 1 independent direction, so has dimension 1.

But what is the dimension of $\mathbb{F}_3^5$? or of $\mathrm{Fun}(\mathbb{Z}, \mathbb{R})$?

To talk about this, we need an algebraic definition of dimension. For this, we have to define bases of vector spaces.

field

Definition: Let V be a vector space over F.
A linear combination of vectors $v_1, \ldots, v_k \in V$ is a vector in V of the form $c_1 v_1 + c_2 v_2 + \cdots c_n v_n$ where each $c_i \in F$.

For $S \subset V$ a subset (not necessarily a subspace), let Span(S) denote the set of all linear combinations of elements of S. (If S is infinite, one only considers finite linear combinations.)

Span(S) is also written as $\langle S \rangle$.

Fact: Span(S) is a subspace of V.

Definition: We say $S \subset V$ is a spanning set if $\mathrm{span}(S) = V$.

Example: Let $S = \{(1,0),(0,1)\} \subset \mathbb{R}^2$.

Any $(x,y) \in \mathbb{R}^2$ can be written as

$x(1,0) + y(0,1)$, so

$\mathrm{Span}(S) = \mathbb{R}^2$, and S is a spanning set.

Exercise: Let $S = \{(1,1),(1,-1)\}$.

Is S a spanning set for $\mathbb{R}^2$?