AMAT 584 Lecture 18, 3/2/20

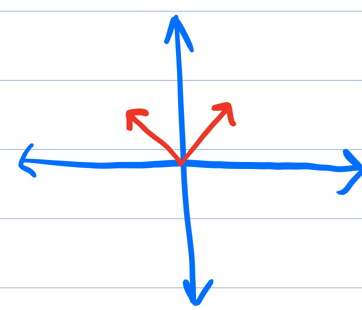
Today: Spanning Set Examples
Linear independence
Bases
Dimension

For $S \subset V$ a subset (not necessarily a subspace), let Span(S) = $\langle S \rangle$ denote the set of all linear combinations of elements of S.

Fact: Span(S) is a subspace of V.

Definition: We say $S \subset V$ is a spanning set if span(S) = V.

Example: Let $S = \{(1,1), (1,-1)\}$.



Is S a spanning set for $\mathbb{R}^2$?

Yes: $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 = \frac{1}{2}(x+y)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(x-y)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Example: Is $S = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix}\right\}$ is a spanning set for $\mathbb{R}^2$?

Yes, e.g.

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 = \frac{1}{2}(x+y)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(x-y)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0\begin{pmatrix} 3 \\ 7 \end{pmatrix}.$$

But clearly, this is not a <u>minimal</u> spanning set for $\mathbb{R}^2$: I can remove $\begin{pmatrix} 3 \\ 7 \end{pmatrix}$ from S, and still have a spanning set.

<u>Definition:</u> A spanning set S is <u>minimal</u> if removing any one element of S yields a subset which is not a spanning set.

<u>Definition:</u> A <u>basis</u> for a vector space V is a minimal spanning set.

<u>Example:</u> $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

More generally, for any $n \geq 1$, field F, and $i \in \{1, \ldots, n\}$, let

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

1 = multiplicative identity of F in the <u>ith</u> position
0 = additive identity elsewhere

Then $B = \{e_1, e_2, \dots, e_n\}$ is a basis for V.

<u>Example</u>: $S = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

- We've already seen that it's a spanning set.
- And clearly, if we remove either vector from S, then we are not left with a spanning set.

<u>Linear independence</u> Let S be a set of vectors in a vector space V over a field F.

<u>Def</u>: S is linearly independent if for all $\vec{v}_1, \dots, \vec{v}_k \in S$

$$\underbrace{c_1\vec{v}_1 + \cdots c_k\vec{v}_k}_{\text{linear combination with coefficients } c_i \in F} = 0 \text{ only if } c_1 = c_2 = \cdots c_k = 0.$$

<u>Example</u>: $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\} \subset \mathbb{R}^2$ is not linearly independent,

e.g. $1\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix} + {}^{-}1\begin{pmatrix}1\\1\end{pmatrix} = 0$

In general, to check directly whether $\{v_1, \dots, v_k\} \subset \mathbb{R}^n$ is linearly independent, we consider the matrix

$$A = (v_1 | v_2 | \cdots | v_k).$$

$\{v_1, \dots, v_k\}$ is linearly independent iff the only solution to $A\vec{x} = \vec{0}$ is $\vec{0}$. (Note: $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = v_1x_1 + v_2x_2 + \cdots + v_nx_n$).

(The standard way to solve this is with Gaussian elimination + backsolve, though to determine whether a non-zero solution exists, one can take a shortcut and just compute rank(A)}.

Example: $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^3$.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Gaussian elimination:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now use backsolve:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} x_1 + x_2 &= 0 \\ x_2 - x_3 &= 0 \\ x_3 &= 0 \end{aligned} \Rightarrow x_1 = x_2 = x_3 = 0$$

So S is linearly independent.

Proposition: A set S of vectors in V is a basis for V if and only if

1. $\mathrm{Span}(S) = V$
2. S is linearly independent.

The above provides an alternative definition of a basis which is perhaps the more common definition.

The proof of the proposition is easy. We will not cover it in class.

Dimension

Proposition: If B and B' are both bases for a vector space V, then there is a bijection

$$f : B \to B'.$$

In particular, if either is finite, then both are, and they have the same number of elts.