AMAT 584 Lecture 19 3/3/20

Today: Dimension
Representing an abstract vector with respect to a basis
Linear maps

Review
Let V be a vector space over a field F.

Def: A basis for V is a minimal spanning set for V.

Equivalently, a basis for V is a linearly independent spanning set.

Proposition: Every vector space has a basis.

Proof uses Zorn's lemma, an axiom from set theory equivalent to the axiom of choice.

Proposition: If B and B' are both bases for a vector space V, then there is a bijection

$f: B \to B'$.

We will not cover these proofs; see Axler's text for details.

Definition: The dimension of a vector space V, denoted $\dim(V)$, the number of elements in a basis for V.

By the two propositions, this number is well defined.

Examples: 1) For any field F,

$$F^n = \{(x_1, \dots, x_n) \mid x_i \in F\}$$

has basis $\{e_1, \dots, e_n\}$, as discussed in the last lecture, so $\dim(F^n) = n$.

2) Let $\vec{v}$ be any non-zero vector in $\mathbb{R}^2$, and the $L = \mathrm{Span}(\{\vec{v}\})$, i.e. L is the line through the origin containing $\vec{v}$.

Then $\{\vec{v}\}$ is a basis for L, so $\dim(L) = 1$.

Thus a line is one-dimensional, as expected.

3) Let S be a finite set, and F be any field.

As mentioned previously, the set $Fun(S,F)$ of all functions $\gamma: S \to F$ is a vector space.

For $x \in S$, let $\delta_x : S \to F$ be given by

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

It is easily checked that $B = \{\delta_x \mid x \in S\}$ is a basis for $Fun(S,F)$.

Clearly there is a bijection between S and B.

Thus, $\dim(Fun(S,F)) = \#$ of elements of S (possibly ∞).

In elementary linear algebra, one usually worries about finite-dimensional vector spaces, and these will suffice for purposes in this course (mostly).

Representing a vector with respect to a basis

Let V be a finite dimensional vector space over the field F.

Given a fixed choice of basis $B = \{b_1, \ldots b_n\}$ for F, we can represent any vector $\vec{v} \in F$ as an element $[\vec{v}]_B \in F^n$.

To explain, we need the following:

Proposition: For $\vec{v}$ and B as above, $\vec{v}$ has a unique expression as a linear combination of elements of B. That is,

$$\vec{v} = c_1 b_1 + \cdots c_n b_n \quad \text{for unique } c_1, \ldots, c_n \in F.$$

Proof: $\mathrm{Span}(B) = V$, so $\vec{v} \in \mathrm{Span}(B)$, so such c_i exist.

If
$$\begin{aligned} \vec{v} &= c_1 b_1 + \cdots + c_n b_n \quad &&\text{for } c_1, \ldots, c_n \in F \\ &= d_1 b_1 + \cdots + d_n b_n &&\phantom{\text{for }} d_1, \ldots, d_n \in F, \end{aligned}$$

then
$$\begin{aligned} \vec{0} &= (c_1 b_1 + \cdots c_n b_n) - (d_1 b_1 + \cdots + d_n b_n) \\ &= (c_1 - d_1) b_1 + \cdots + (c_n - d_n) b_n. \end{aligned}$$

B is linearly independent, so $c_i - d_i = 0 \ \forall i$, i.e. $c_i = d_i$. This gives uniqueness. ■

$$\boxed{\text{We define } [v]_B = (c_1, \dots, c_n) \in F^n.}$$

<u>Example</u>: Let $V = \mathbb{R}^2$, $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

Last lecture, we observed that $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}(x+y)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(x-y)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

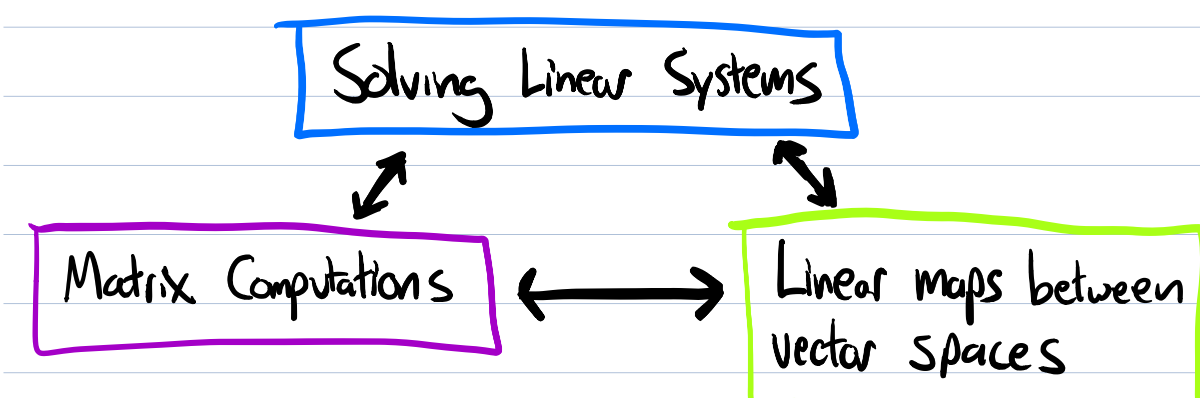
So $\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_B = \begin{pmatrix} \frac{1}{2}(x+y) \\ \frac{1}{2}(x-y) \end{pmatrix}$

For example, $\left[\begin{pmatrix} 7 \\ 3 \end{pmatrix}\right]_B = \begin{pmatrix} \frac{1}{2}(10) \\ \frac{1}{2}(4) \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

The basis B gives an <u>alternative coordinate system</u> on $\mathbb{R}^2$.

<u>Linear Maps</u>

Linear algebra studies the relation between three closely related things:



In elementary courses, linear maps are emphasized less, but understanding their connection to linear systems and matrices can be very clarifying.

In any case, we will need linear maps to talk about homology.

<u>Definition</u>: A function of vector spaces $f: V \to W$, both over F, is said to be linear if:

$$f(\vec{v}+\vec{w}) = f(\vec{v})+f(\vec{w}) \quad \forall\, \vec{v},\vec{w} \in \vec{V}$$

$$f(c\vec{v}) = c f(\vec{v}) \quad \forall\, \vec{v} \in \vec{V},\ c \in F.$$

<u>Example</u>: any $m \times n$ matrix A with coefficients in a field F defines a linear map $T^A : \mathbb{R}^n \to \mathbb{R}^m$ given by

$$T^A(x) = \underbrace{Ax}_{\text{matrix-vector multiplication}}.$$