Today: Linear maps

Definition: Let V, W be vector spaces over a field K. $f: V \to W$ is said to be *linear* if:

$$f(\vec{v}+\vec{w}) = f(\vec{v}) + f(\vec{w}) \quad \forall\, \vec{v}, \vec{w} \in \vec{V}$$

$$f(c\vec{v}) = c f(\vec{v}) \quad \forall\, \vec{v} \in \vec{V},\ c \in F.$$

We call such f a linear map or linear transformation.

Example: any $m \times n$ matrix A with coefficients in a field F defines a linear map $T^A: \mathbb{R}^n \to \mathbb{R}^m$ given by

$$T^A(\vec{x}) = \underbrace{A\vec{x}}_{\text{matrix-vector multiplication}}.$$

For instance, if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

then $T^A: \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T^A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \\ 7x_1 + 8x_2 + 9x_3 \end{pmatrix}$$

The linearity of T^A follows from basic properties of matrix-vector multiplication

Subspaces associated to a linear map $f: V \to W$:

$\mathrm{im}(f) = \{\vec{w} \in W \mid \vec{w} = f(\vec{v}) \text{ for some } \vec{v} \in V\}$.

$\ker(f) = \{\vec{v} \in V \mid f(\vec{v}) = 0\}$.

called the kernel of f.

Proof that $\mathrm{im}(f)$ is a subspace:

- If $\vec{w}, \vec{w}' \in \mathrm{im}(f)$, then $\vec{w} = f(\vec{v})$ and $\vec{w}' = f(\vec{v}')$ for some $\vec{v}, \vec{v}' \in V$. By linearity $f(\vec{v} + \vec{v}') = f(\vec{v}) + f(\vec{v}') = \vec{w} + \vec{w}'$, so $\vec{w} + \vec{w}' \in \mathrm{im}(f)$.
- If $c \in k$, then $f(c\vec{v}) = cf(\vec{v}) = c\vec{w}$, so $c\vec{w} \in \mathrm{im}(f)$. ∎

The proof that $\ker(f)$ is a subspace is quite similar. I leave it as an exercise.

Definitions For $f: V \to W$ a linear map:

1. We call $\dim(\mathrm{im}\, f)$ the rank of f.
2. We call $\dim(\ker f)$ the nullity of f.

Rank-Nullity Theorem: For $f: V \to W$ a linear map between finite-dimensional vector spaces,

$$\dim(V) = \mathrm{rank}(f) + \mathrm{nullity}(f).$$

How does one compute rank(f) or nullity(f) in practice? We will address this below.

Representing Linear Maps with Matrices

Recall from last lecture: For V a finite dimensional vector space over F with basis $B=\{b_1,\dots,b_m\}$ and $\vec{v}\in V$, we can write

$$\vec{v}=c_1 b+\cdots+c_m b_m \text{ for unique } c_1,\dots,c_m\in F.$$

We call this the representation of $\vec{v}$ in the basis B.

We let $[\vec{v}]_B=\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}\in F^m$

Let $f:V\to W$ be a linear map of finite dimensional vector spaces.

Given bases $B=\{b_1,\dots,b_n\}$ and $B'=\{b'_1,\dots,b'_m\}$ for V and W respectively, we can represent f via the $\underset{\#\text{rows}}{m}\times\underset{\#\text{columns}}{n}$ matrix

$$[f]_{B',B}=\big(\underset{1^{st}\text{ column}}{[f(b_1)]_{B'}}\,\big|\,\underset{2^{nd}\text{ column}}{[f(b_2)]_{B'}}\,\big|\cdots\big|\,\underset{n^{th}\text{ column}}{[f(b_n)]_{B'}}\big)$$

In words, the j^{th} column of $[f]_{B',B}$ is the representation of $f(b_j)$ in the basis B'.

The idea behind this is that by linearity, to know the value of f on an arbitrary $\vec{v} \in V$, it's enough to know the value of $f(b)$ for each $b \in B$.

Indeed, if $\vec{v} = c_1 b_1 + \cdots c_n b_n$, then by linearity
$$f(\vec{v}) = f(c_1 b_1 + \cdots + c_n b_n) = c_1 f(b_1) + \cdots + c_n f(b_n).$$

Fact: If S is the set of columns of $[f]_{B',B}$, then

$$\text{Rank}(f) = \dim(\text{Span}(S)).$$

This can be computed by putting $[f]_{B',B}$ into row echelon form using Gaussian Elimination: The number of pivot rows in the reduced matrix is the rank of f.

Example: Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}, \text{ ie } f = T^A, \text{ where } A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let $B = \left\{ \underset{b_1}{\underset{\parallel}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}}, \underset{b_2}{\underset{\parallel}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \right\}$, $B' = \left\{ \underset{b_1'}{\underset{\parallel}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}}, \underset{b_2'}{\underset{\parallel}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}} \right\}$

$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = 1\begin{pmatrix}1\\1\end{pmatrix} + 0\begin{pmatrix}1\\-1\end{pmatrix}$, so $[f(b_1)]_{B'} = \begin{pmatrix}1\\0\end{pmatrix}$

$f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = 0\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}1\\-1\end{pmatrix}$, so $[f(b_2)]_{B'} = \begin{pmatrix}0\\1\end{pmatrix}$.

$[f]_{B,B_1} = \left([f(b_1)]_{B'} \mid [f(b_2)]_{B'}\right) = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ is already in reduced echelon form. It has two pivots, so rank$(f) = 2$.

By the rank-nullity thm, nullity$(f) = 2 - 2 = 0$.