AMAT 584 Lecture 21 3/9/20

Today: Power Sets as Vector Spaces
Simplicial Chain complexes

We've now covered most of the linear algebra we will need to define homology, except quotient spaces.

Instead of covering quotient spaces now, we'll turn to the linear algebra of chain complexes, and then consider quotient spaces after.

Power Sets as Vector Spaces

For S a set, let $P(S)$ denote the set of all subsets of S. This is called the power set of S, and is sometimes denoted 2^S.

Let $S=\{A,B\}$. $P(S)=\{\emptyset, \{A\}, \{B\}, S\}$.

Note: If S is finite, then $P(S)$ has $2^{|S|}$ elements.

Recall that $\text{Fun}(S,\mathbb{F}_2)$ denotes the vector space of all functions $f:S\to\mathbb{F}_2$.

There is a natural correspondence between $P(S)$ and $Fun(S, \mathbb{F}_2)$:

Define a function $\gamma: P(S) \to Fun(S, \mathbb{F}_2)$ by

$$\gamma(T)(x) = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{otherwise} \end{cases}$$

To keep our notation nice, let's write $\gamma(T)$ as γ^T.

<u>Example</u>: $S = \{A, B, C\}$ $T = \{B, C\}$

$\gamma^T : S \to \mathbb{F}_2$ is given by

$$\gamma^T(x) = \begin{cases} 0 & \text{if } x = A \\ 1 & \text{if } x = B \text{ or } x = C. \end{cases}$$

<u>Proposition</u>: For all sets S, $\gamma : P(S) \to Fun(S, \mathbb{F}_2)$ is a bijection.

Proof: γ is invertible:

$$\gamma^{-1}(f) = \{x \in S \mid f(x) = 1\}.$$

Hence it's a bijection.

By way of this bijection, we can think of vectors in $\mathrm{Fun}(S, \mathbb{F}_2)$ as subsets of S.

Note: $\gamma(\emptyset) = \vec{0}$, i.e., the empty subset corresponds to the additive identity in $\mathrm{Fun}(S, \mathbb{F}_2)$.

How do we understand addition and scalar multiplication from this viewpoint?

Well, in any vector space V over $\mathbb{F}_2$, scalar multiplication is not to interesting:

$$1\vec{v} = \vec{v} \quad \forall\ \vec{v} \in V \quad \text{(that's an axiom)}$$
$$0\vec{v} = \vec{0} \quad \forall\ \vec{v} \in V \quad \text{(this follows easily from the axioms).}$$

Addition is more interesting:

For any sets A and B, let the symmetric difference of A and B be the set

$$SD(A,B) = A \cup B - A \cap B = \{x \in A \cup B \mid x \notin A \cap B\}.$$

Note: Given any set S, symmetric difference defines a function (i.e., "operator")

$$SD: P(S) \times P(S) \to P(S).$$

Fact: Under the correspondence γ between $P(S)$ and $Fun(S, F_2)$, the operator SD on $P(S)$ corresponds to addition in $Fun(S, F_2)$.

More formally, $\forall\, T, U \subset S$, $\gamma^T + \gamma^U = \gamma^{SD(T,U)}$

This is easy to prove. We'll just illustrate the idea with an example.

Example:

Let $S = \{A, B, C\}$ $\quad T = \{B, C\} \subset S$

$U = \{A, B\} \subset S$.

As seen above,

$\gamma^T : S \to F_2$ is given by

$\gamma^T(A) = 0$
$\gamma^T(B) = 1$
$\gamma^T(C) = 1$

$\gamma^T : S \to F_2$ is given by

$\gamma^U(A) = 1$
$\gamma^U(B) = 1$
$\gamma^U(C) = 0$

$\gamma^T + \gamma^U : S \to \mathbb{F}_2$ is given by

$$\gamma^{T+U}(A) = 0+1 = 1$$
$$\gamma^{T+U}(B) = 1+1 = 0$$
$$\gamma^{T+U}(C) = 1+0 = 1$$

$$SD(T,U) = \{A, C\}$$

$\gamma^T : S \to \mathbb{F}_2$ is given by

$$\gamma^T(A) = 1$$
$$\gamma^T(B) = 0$$
$$\gamma^T(C) = 1,$$

so we indeed have that $\gamma^T + \gamma^U = \gamma^{SD(T,U)}$.

<u>Formal linear combinations</u>

For any set S and field F, we define a subspace

$FLC(S,F) < Fun(S,F)$ by

$$FLC(S,F) = \{ f \in Fun(S,F) \mid f(x) \neq 0 \text{ for only finitely many } x \}.$$