# AMAT 584 Lecture 22 3/23/20

**Today:** More about linear maps (isomorphisms)
Simplicial Chain Complexes

## More About Linear Maps

It would have been more natural to cover this before lecture 21 on power sets as vector spaces, but I forgot to.

Recall the following definition from Lec. 20:

**Definition:** Let $V, W$ be vector spaces over a field $K$. $f: V \to W$ is said to be linear if:

$$f(\vec{v}+\vec{w}) = f(\vec{v})+f(\vec{w}) \quad \forall\, \vec{v},\vec{w} \in \vec{V}$$

$$f(c\vec{v}) = c f(\vec{v}) \quad \forall\, \vec{v} \in \vec{V},\ c \in K.$$

We call such $f$ a linear map or linear transformation.

**Recall:** If $f: V \to V'$ is a linear map between finite-dimensional vector spaces with $B$ an ordered basis for $V$ and $B'$ an ordered basis for $V'$, then we can represent $f$ via a matrix $[f]_{B,B'}$ whose $j^{th}$ column represents $f(B_j)$ as a linear combination of the $B'$.

[see lecture 20 for details]

**Facts:** 1. For any vector space $V$, $Id_V : V \to V$ is a linear map.

2. If $f : V \to W$ and $g : W \to X$ are linear maps, then $g \circ f : W \to X$ is a linear map.

3. If, $A, B, C$ are bases for $V, W, X$, respectively, then $[g \circ f]_{C,A} = \underbrace{[g]_{C,B}[f]_{B,A}}_{\text{matrix multiplication.}}$.

**Recall:** If $M$ is an $a \times b$ matrix and
$N$ is a $b \times c$ matrix,
the matrix product $MN$ is the $a \times c$ matrix whose $(i,j)^{th}$ entry is the product of $i^{th}$ row of $M$ with the $j^{th}$ column of $N$, i.e., $(MN)_{ij} = \sum_{k=1}^{b} M_{ik} N_{kj}$.

## Isomorphisms

A bijective linear map is called an isomorphism.

**Note:** If $f : V \to W$ is an isomorphism, then since it is a bijection it has an inverse $f^{-1} : W \to V$. It is easily checked that $f^{-1}$ is also linear.

If there exists an isomorphism $f : V \to W$, we say $V$ and $W$ are isomorphic.

Intuitively, isomorphic vector spaces have the same algebraic structure

Examples:

1. For any vector space $V$, $Id_V : V \to V$ is an isomorphism.
2. Let $V \subset \mathbb{R}^2$ be the line

$$V = \{(x,y) \mid y = x\}$$

Then $f : \mathbb{R} \to V$ given by
$f(x) = (x,x)$ is an isomorphism.

3. Let $S = \{1, \dots, n\}$ and let $F$ be any field.
(for concreteness, you may think of the case $F = \mathbb{R}$).

Recall that $Fun(S,F)$ denotes the vector space of all functions from $S$ to $F$.

Define a function $f : Fun(S,F) \to F^n$ by
$f(\gamma) = (\gamma(1), \gamma(2), \gamma(3), \dots, \gamma(n))$.

$f$ is an isomorphism.

4. [This example was implicit in lecture 21.]
For any set $S$, $\underbrace{P(S)}_{\text{set of all subsets of } S}$ is a vector space over $F_2$,

with addition given by symmetric difference.

The function $\gamma: P(S) \to \mathrm{Fun}(S, \mathbb{F}_2)$ given by

$$\gamma^T(x) = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{otherwise} \end{cases}$$

recall: We write $\gamma^T$ for $\gamma(T)$

considered in Lec 21 is an isomorphism.

Proposition: Two finite-dimensional vector spaces over the same field are isomorphic if and only if they have the same dimension.

The proof is an easy exercise.

[This does extend easily to infinite-dimensional vector spaces, but I will not bother with that.]

Example: Let $S = \{1, \ldots, n\}$.

Then $\mathrm{Fun}(S, \mathbb{F}_2)$, $P(S)$, and $\mathbb{F}_2^n$ all have the same dimension, namely $n$.

# Chain Complexes

- Constuction involved in the definition of homology
- Idea: Convert the set-theoretic structure of the simplicial complex into linear algebra.

The construction of simplicial chain complexes (and of homology) requires a choice of field.

To keep things as simple as possible, I will work only with $F_2$. This is by far the most common choice in TDA.

Let $X$ be a finite simplicial complex. Assume the vertex set $V(X)$ is ordered.

For $j \geq 0$, let $X^j$ denote the set of $j$ simplices.

Let $C_j(X) = Pow(X^j)$, regarded as a vector space over $F_2$.

We call $C_j(X)$ the $j^{th}$ chain vector space of $X$. Elements of $C_j(X)$ are called chains.

**Notation:** $\{\sigma_1, \dots, \sigma_k\} \in C_j(X)$ is written as

$$\sigma_1 + \sigma_2 + \dots + \sigma_n.$$

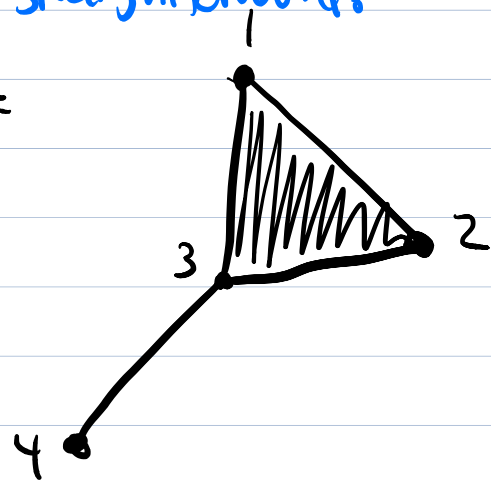
In particular, we write $\{\sigma\} \in C_j(X)$ simply as $\sigma$. This identifies $X^j$ with a subset of $C_j(X)$.

We write $\{\} \in C_j(X)$ as $\vec{0}$, since it's the additive identity.

__Proposition:__ $X^j$ is a basis for $C_j(X)$.

The proof is straightforward.

__Example:__ $X =$




$X^0 = \{[1], [2], [3], [4]\}$,

$X^1 = \{[1,2], [1,3], [2,3], [3,4]\}$.

$X^2 = \{[1,2,3]\}$.

$C_0(X)$ consists of all possible subsets of $X^0$.

in our notation, $[1] \in X$, $'[1]+[2]+[3] \in X$, for example.

$X^0 = \{[1],[2],[3],[4]\}$ is a basis for $C_0(X)$.

Next time, we'll define a sequence of linear maps

$$\xrightarrow{\delta_{j+1}} C_j(X) \xrightarrow{\delta_j} C_{j-1}(X) \xrightarrow{\delta_{j-1}} \cdots \xrightarrow{\delta_2} C_1(X) \xrightarrow{\delta_1} C_0(X).$$

This is the chain complex of X.