AMAT 584 Lec 23 3/25/20

Today: Chain Complexes, Continued

We started the discussion of chain complexes in the last lecture, but didn't get too far. We'll start by reviewing that material.

Chain Complexes

Given a finite abstract simplicial complex X, we construct a sequence of vector spaces over F_2 and linear maps:

$$\cdots \xrightarrow{\delta_3} C_2(X) \xrightarrow{\delta_2} C_1(X) \xrightarrow{\delta_1} C_0(X)$$

The maps δ_j are called boundary maps, for reasons that will become clear later.

Techically, this sequence is infinite, but $C_j(X)$ is the trivial vector space whenever $j > \dim(X)$, so the interesting part of the sequence is finite.

Recall: X^j denotes the set of j-simplices in X.

$$C_j(X) = P(X^j)$$

power set = set of all subsets

$C_j(X)$ is a vector space over F_2, with + the symmetric difference operator.

Notation: $\{\sigma_1, \sigma_2, \dots, \sigma_k\} \in C_j(X)$ is written as $\sigma_1 + \sigma_2 + \cdots + \sigma_n$.

This is not a crazy convention, since in fact
$\{\sigma_1, \dots, \sigma_k\} = \{\sigma_1\} + \{\sigma_2\} + \cdots + \{\sigma_k\}$.
We're just dropping some curly brackets.

In particular, we write $\{\sigma\} \in C_j(X)$ simply as σ.

As this notation suggests, we can identify X^j with a subset of $C_j(X)$, namely the subset of singleton sets.

Fact: X^j is a basis for $C_j(X)$.
Thus $\dim(C_j(X)) = |X^j| = \#$ j-simplices of X.
The proof is an easy exercise.

The additive identity of $C_j(X)$ is $\{\}$. In keeping with our broader conventions, this is denoted $\vec{0}$.

Example $X = \{[1], [2], [1,2]\}$.

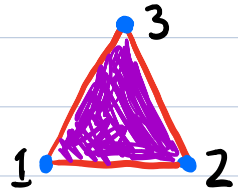
$$C_0(X) = \{\underset{\{\}}{\underset{\shortparallel}{\vec{0}}}, \underset{\{[1]\}}{\underset{\shortparallel}{[1]}}, \underset{\{[2]\}}{\underset{\shortparallel}{[2]}}, \underbrace{[1]+[2]}_{\{[1],[2]\}}\}$$

$\{[1],[2]\}$ is a basis for $C_0(X)$

$C_1(X) = \{\vec{0}, [1,2]\}$.
$\{[1,2]\}$ is a basis for $C_1(X)$.

$C_j(X) = \{\vec{0}\}$ for $j \geq 2$. We say that $C_j(X)$ is trivial.

Example: $X = \{[1],[2],[3],[1,2],[2,3],[1,3],[1,2,3]\}$.



$\{[1],[2],[3]\}$ is a basis for $C_0(X)$.
$\{[1,2],[2,3],[1,3]\}$ is a basis for $C_1(X)$.
$\{[1,2,3]\}$ is a basis for $C_2(X)$.

Boundary Maps

Notation: For $[x_0, \dots, x_j, \dots, x_k] \in X^k$, let $[x_0, \dots, \hat{x}_j, \dots, x_k] \in X^{k-1}$ be the simplex obtained by removing x_j.
This is sometimes called a facet of $[x_0, \dots, x_j, \dots, x_k]$.

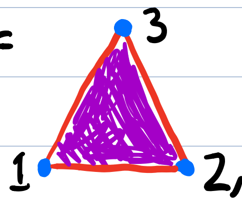
Example: In the triangle example above,

$[\hat{1},2,3] = [2,3]$	$[\hat{1},2] = [2]$
$[1,\hat{2},3] = [1,3]$	$[1,\hat{2}] = [1]$.
$[1,2,\hat{3}] = [1,2]$	

For $\sigma = [x_0, \dots, x_j, \dots, x_k] \in X^k$, we define the **boundary** of σ, denoted $\delta(\sigma)$, by

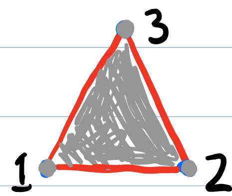
$$
\begin{aligned}
\delta(\sigma) = {} & [x_1, x_2, \dots, x_k] && \leftarrow x_0 \text{ removed} \\
& + [x_0, x_2, x_3, \dots, x_k] && \leftarrow x_1 \text{ removed} \\
& + [x_0, x_1, x_3, x_4, \dots, x_k] && \leftarrow x_2 \text{ removed} \\
& + \\
& \vdots \\
& + [x_0, x_1, \dots, x_{k-1}] && \leftarrow x_k \text{ removed} \\
= {} & \sum_{j=0}^{k} [x_0, \dots, \hat{x}_j, \dots, x_k] \in C_{k-1}(X).
\end{aligned}
$$

Example: For $X =$



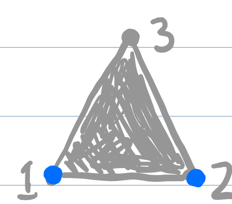
$\delta([1,2,3]) = [2,3] + [1,3] + [1,2]$.

Illustration (in red):



$\delta([1,2]) = [2] + [1]$.

Illustration (in blue):

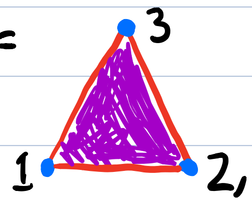


Now, for $j \geq 1$, we define $\delta_j : C(X) \to C_{j-1}(X)$ by

$$\delta_j(\sigma_1 + \sigma_2 + \cdots + \sigma_k) = \delta(\sigma_1) + \delta_2(\sigma_2) + \cdots + \delta(\sigma_k).$$

We define $C_{-1}(X)$ to be a trivial vector space over F_2 and define $\delta_0 : C_0(X) \to C_{-1}(X)$ to be the trivial map.

<u>Example</u>: For $X =$



$$\delta_2([1,2,3]) = \delta([1,2,3]) = [2,3] + [1,3] + [1,2].$$

More interestingly,

$$\delta_1([2,3] + [1,3] + [1,2])$$

$$= [2] + [3] + [1] + [3] + [1] + [2]$$

$$= ([1] + [1]) + ([2] + [2]) + ([3] + [3])$$

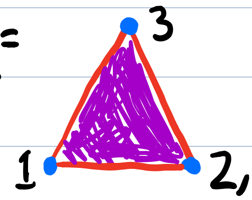
$$= \vec{0} + \vec{0} + \vec{0} = \vec{0}.$$

(since the symmetric difference of a set with itself is the empty set.)

Proposition: Each δ_j is a linear map.

Thus, if each X^j is ordered, we can represent $\delta_j : C_j(X) \to C_{j-1}(X)$ with respect to the bases X_j and X_{j-1} as a matrix.

Example: Consider $X =$



Consider the following orderings of X^0 and X^1

X^0: [1], [2], [3]

X^1: [2,3], [1,3], [1,2]

δ_2 is represented by the 3×1 matrix:

$$\begin{array}{cc} & [1,2,3] \\ \begin{array}{c} [2,3] \\ [1,3] \\ [1,2] \end{array} & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{array}$$

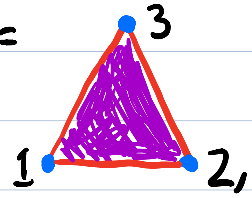
δ_1 is represented by the 3×3 matrix:

$$\begin{array}{cc} & \begin{array}{ccc} [2,3] & [1,3] & [1,2] \end{array} \\ \begin{array}{c} [1] \\ [2] \\ [3] \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}.$$

I've labeled the rows and collumns of the matrices by the corresponding basis elements

Proposition: For all $j \geq 1$, $\delta_{j-1} \circ \delta_j = 0$.

Example: A calculation given above shows that for

$X =$ 

$\delta_1 \circ \delta_2 = 0$. In the calculation, we see that the simplices of $\delta_1 \circ \delta_2([1,2,3])$ cancel in pairs.

More generally, the proof of the proposition amounts to the observation that for any $\sigma \in X^j$, the simplices of $\delta_1 \circ \delta_2(\sigma)$ cancel in pairs, in the same way.

See the course references for details.