AMAT 584 Lecture 24, 3/27/20

<u>Today</u>: Chain Complexes, continued
Quotient Vector Spaces

To start, let's review the definition of a chain complex:

For X a finite simplicial complex, the <u>chain complex</u> of X is the sequence of vector spaces and linear maps

$$\cdots \xrightarrow{\delta_4} C_3(X) \xrightarrow{\delta_3} C_2(X) \xrightarrow{\delta_2} C_1(X) \xrightarrow{\delta_1} C_0(X) \xrightarrow{\delta_0} 0$$

Where $C_j(X) = \underbrace{P}_{\text{power set}}(X^j)$, for X^j the set of j-simplices of X,
power set, regarded as vector space over F_2.

$$\delta : X^j \to C_{j-1}(X), \quad \delta([x_0, \ldots, x_j]) = \sum_{k=0}^{j} [x_0, \ldots, \hat{x}_k, \ldots, x_j]$$

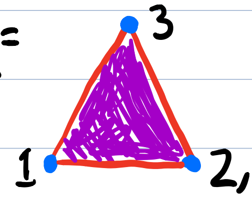
$$\delta_j(\sigma_1 + \cdots + \sigma_k) = \delta(\sigma_1) + \delta(\sigma_2) + \cdots + \delta(\sigma_n).$$

<u>Recall</u>: X^j is (identified with) a basis for $C_j(X)$, called the standard basis.

$$\Rightarrow \dim(C_j(X)) = |X^j|.$$

Since each $C_j(X)$ is finite dimensional, if we order $C_j(X)$ and $C_{j-1}(X)$, then we can represent δ_j via a matrix with coefficients in F_2.

<u>Example</u>: Consider $X=$



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Consider the following orderings of X^0 and X^1

X^0: [1], [2], [3]

X^1: [2,3], [1,3], [1,2]

δ_2 is represented by the 3×1 matrix:

$$\begin{array}{c c} & [1,2,3] \\ \begin{matrix}[2,3]\\ [1,3]\\ [1,2]\end{matrix} & \begin{pmatrix}1\\1\\1\end{pmatrix}\end{array}$$

δ_1 is represented by the 3×3 matrix:

$$\begin{array}{c c} & \begin{matrix}[2,3] & [1,3] & [1,2]\end{matrix} \\ \begin{matrix}[1]\\ [2]\\ [3]\end{matrix} & \begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\end{array}.$$

I've labeled the rows and collumns of the matrices by the corresponding basis elements

Note that $\delta_1 \circ \delta_2 = \begin{pmatrix}0+1+1\\ 1+0+1\\ 1+1+0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$

More generally, we have the following

Proposition: $\forall\, j \geq 0$,

$$\delta_j \circ \delta_{j+1} : C_{j+1}(X) \to C_{j-1}(X) = 0,$$

i.e., is the constant map to $\vec{0}$.

Example: For X as above,

$$\begin{aligned}
\delta_1 \circ \delta_2([1,2,3]) &= \delta_1(\delta([1,2,3])) \\
&= \delta_1([2,3]+[1,3]+[1,2]) \\
&= \delta(2,3)+\delta(1,3)+\delta(1,2) \\
&= [2]+[3]+[1]+[3]+[1]+[2] \quad * \\
&= ([1]+[1])+([2]+[2])+([3]+[3]) \\
&= \vec{0}+\vec{0}+\vec{0} = \vec{0}.
\end{aligned}$$

Note that $\delta_1 \circ \delta_2([1,2,3])$ is $\vec{0}$ because the $\vec{0}$ simplices in the sum $*$ cancel in pairs.
The proof of the proposition amounts to the observation that this phenomena generalizes to any simplex in any simplicial complex.

Cycles and Boundaries

For $j \geq 0$,

$\ker(\delta_j) \subset C_j(X)$ is called the cycle subspace, and is denoted $Z_j(X)$. Elements of $Z_j(X)$ are called j-cycles.

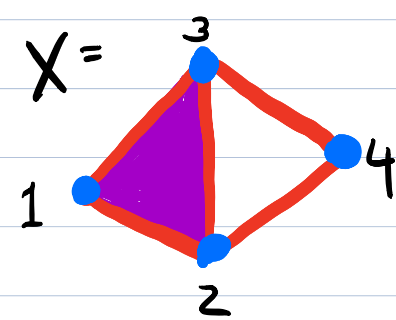
$\operatorname{im}(\delta_{j+1}) \subset C_j(X)$ is called the image subspace, and is denoted $B_j(X)$. Elements of $B_j(X)$ are called j-boundaries.

Proposition: $B_j(X) \subset Z_j(X)$.

Proof: If $z \in B_j(X) = \operatorname{im}(\delta_{j+1})$ then $z = \delta_{j+1}(y)$ for some $y \in C_{j+1}(X)$. Then $\delta_j(z) = \delta_j(\delta_{j+1}(y)) = \vec{0}$, so $z \in \ker(\delta_j) = Z_j(X)$. ∎

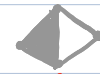
As the names "cycles" and "boundaries" suggest, these objects have a nice geometric interpretation:

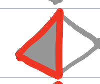
Example: Let $X =$

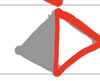


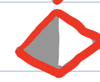
$Z_0(X) = C_0(X)$, since $\delta_0 = 0$.

It can be checked that

$$Z_1(X) = \{\vec{0},$$ 

$$[1,2]+[1,3]+[2,3],$$ 

$$[2,3]+[3,4]+[2,4],$$ 

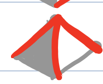
$$[1,2]+[2,4]+[3,4]+[1,3].\}$$ 

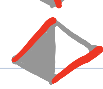
For example, $\delta_1([1,2]+[2,4]+[3,4]+[1,3]) =$

$$[1]+[2]+[2]+[4]+[3]+[4]+[1]+[3]$$
$$=([1]+[1])+([2]+[2])+([3]+[3])+([4]+[4])$$
$$=\vec{0}.$$

Examples of 1-chains which are not 1-cycles include:

$[1,3]+[3,4]+[2,4]$ 

$[1,3]+[3,4]+[2,3]$ 

$[1,3]+[2,4]$ 

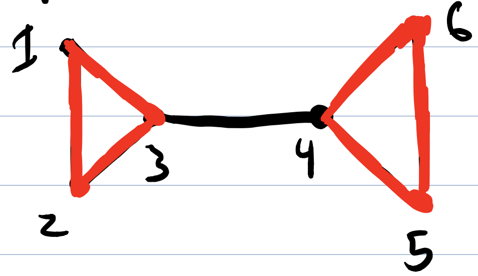
It is easy to see that $B_1(X) = \{\vec{0}, [1,2]+[1,3]+[2,3]\}$.

Indeed, $C_2(X) = \{\vec{0}, [1,2,3]\}$.

$$\delta_2(\vec{0}) = \vec{0} \text{ and } \delta_2([1,2,3]) = \underbrace{[1,2]+[1,3]+[2,3]}_{\text{boundary of a triangle!}}.$$

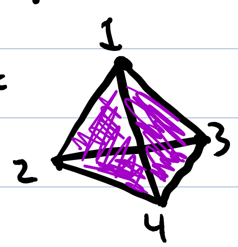
For $j \geq 2$, $Z_j(X) = B_2(X) = \{\vec{0}\}$.

Example: Cycles needn't form a connected subgraph:

$X =$ 

$$[1,2]+[2,3]+[1,3]+[4,5]+[5,6]+[4,6] \in B_1(X).$$

Example: Consider the 3-simplex

$X =$ 

$$\underbrace{[1,2,3]+[2,3,4]+[1,2,4]+[1,3,4]}_{\text{Sum of all the 2-simplices, i.e. hollow tetrahedron.}} \in Z_2(X)$$