# AMAT 584 Lec 25

**Today:** More examples of cycles + boundaries
Quotient Spaces

## Review of Cycles and Boundaries

For $j \geq 0$,

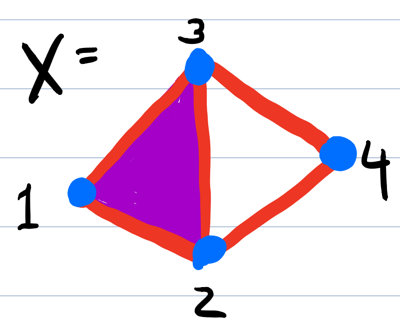
$\ker(\delta_j) \subset C_j(X)$ is called the cycle subspace, and is denoted $Z_j(X)$. Elements of $Z_j(X)$ are called $j$-cycles.

$\operatorname{im}(\delta_{j+1}) \subset C_j(X)$ is called the image subspace, and is denoted $B_j(X)$. Elements of $B_j(X)$ are called $j$-boundaries.

**Proposition:** $B_j(X) \subset Z_j(X)\ \forall j \geq 0$.

**Example from last lecture:**

Let $X =$




$Z_0(X) = \ker(\delta_0) = C_0(X)$
because $\delta_0 : C_0(X) \to 0$
is $0$.

What is $B_0(X) = \mathrm{im}(\delta_1)$?

not necessarily a subspace.

**Fact:** For any linear map $f: V \to W$ and set $S \subset V$, we have $\mathrm{Span}(f(S)) = f(\mathrm{Span}(S))$.

[recall: for $g: A \to B$ any function and $C \subset A$, $g(C)$ is defined by

$$g(C) = \{y \in B \mid y = g(x) \text{ for some } x \in A\}.]$$

$$B_0(X) = \mathrm{im}(\delta_1) = \delta_0(C_1(X)) = \delta_0(\mathrm{Span}(X^1)) = \mathrm{Span}(\delta_0(X^1)).$$

because $X^1$ is a basis for $C_1(X)$

by the fact

$$\delta_0(X^1) = \{[1]+[2], [1]+[3], [2]+[3], [3]+[4], [2]+[4]\}.$$

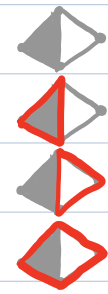
It's easy to check using linear algebra (or by brute force) that $\{[1]+[2], [2]+[3], [3]+[4]\}$ is a basis for $\mathrm{im}(\delta_1) = \mathrm{Span}(\delta_0(X^1))$.

Note, e.g., that $[1]+[3] = ([1]+[2]) + ([2]+[3])$
$[2]+[4] = ([2]+[3]) + ([3]+[4])$.

Thus, $\dim(B_0(X)) = 3$.

We saw last time that

$$Z_1(X) = \{ z_1 = \vec{0},$$
$$z_2 = [1,2] + [1,3] + [2,3],$$
$$z_3 = [2,3] + [3,4] + [2,4],$$
$$z_4 = [1,2] + [2,4] + [3,4] + [1,3]. \}$$



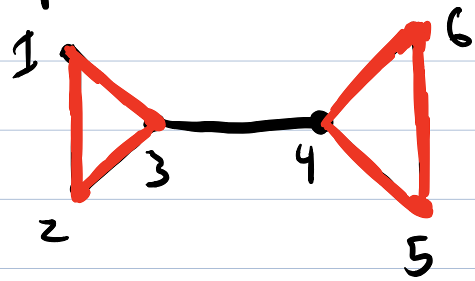
It's easy to check that $\{z_2, z_3\}$ is a basis for $Z_1(X)$.

Note, e.g., that $z_4 = z_2 + z_3$.

As the above example suggests, "closed loops" in the 1-skeleton are 1-cycles.

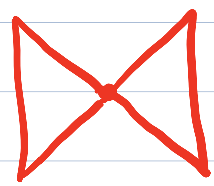
[To make this more precise, recall our TDA I definition of cycles in a graph (Lecture 21 from Fall 2019). Any such cycle is a 1-cycle. But not all 1-cycles are cycles in that sense.]

**Example**: Cycles needn't form a connected subgraph:

$X =$ 

$[1,2]+[2,3]+[1,3]+[4,5]+[5,6]+[4,6] \in B_1(X)$.

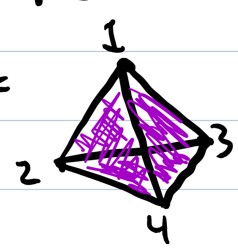
**Example:**



is also a 1-cycle.

**Example:** Consider the 3-simplex

$X =$

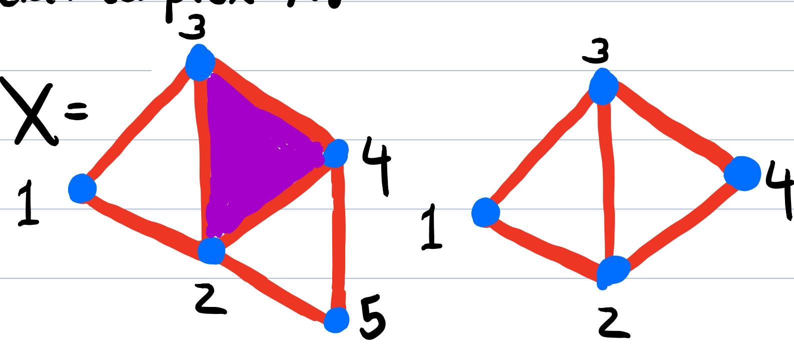



$$\underbrace{[1,2,3]+[2,3,4]+[1,2,4]+[1,3,4]}_{\text{Sum of all the 2-simplices, i.e. hollow tetrahedron.}} \in Z_2(X)$$

## The Idea of Homology

The (initial) goal: For fixed $j \geq 0$, count the $j$-dim holes in a simplicial complex $X$.

**Example** $X =$



According to common usage of the word "hole," $X$ has two holes.

These are holes you can "see through," so should be 1-D holes, according to things we said earlier this semester.

How do we make precise the idea that $X$ has two 1-D holes?

<u>Naive idea 1</u>: # $j$-D holes in $X$ = # elements of $Z_j(X)$.

This is no good.

Here $Z_1(X)$ has 4 elements, but two holes.
One of those elements is 0, which clearly doesn't correspond to a hole.
But even if we consider only non-zero elements of $Z_1(X)$, we get a count of 3, which is still too many.

Intuitively, the issue is that the cycle $z_4 = [1,2]+[2,4]+[3,4]+[1,3]$ is an "extra hole."

If we've already counted $z_2 = [1,2]+[2,3]+[1,3]$ and
$z_3 = [2,3]+[3,4]+[2,4]$,

we don't want to also count $z_4$.

The solution lies in the observation that there is an algebraic relation between these cycles. Indeed, we've seen that $z_4 = z_2 + z_3$.

This motivates the following:

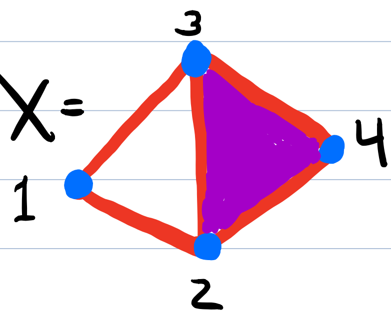
<u>Naive idea 2</u>: # $j$-D holes in $X = \dim(Z_j(X))$.

For graphs, $\dim(Z_1(X))$ is indeed a "correct" way to count the # of holes.

However, for general simplicial complexes, this idea is problematic:

For example, consider

$X =$




$X$ has one 1-D hole but $\dim(Z_1(X)) = 2$.

The problem, intuitively, is that the cycle

$$z_3 = [2,3]+[3,4]+[2,4]$$

is filled in by a triange.

That is, $z_3 \in B_1(X) = \mathrm{im}(\partial_2)$. So it does not contribute a hole.

To account for this kind of thing, we will to modify the vector space $Z_1(X)$ to "remove" the cycles which are boundaries.

To do this, we will use quotient spaces, to be introduced next time.