AMAT 584 Lecture 26 4/1/2020

Today: Quotient Spaces and Homology

Quotient Spaces are the last main linear algebra ingredient we need to define homology.

Given a vector space V over a field F and a subspace $W \subset V$, we define the quotient space V/W to be a vector space over F.

very roughly speaking, V/W is obtained from V by setting elements of W to 0.

If V is finite dimensional, then

$$\dim(V/W) = \dim(V) - \dim(W).$$

Before defining quotient spaces, we describe the application to homology.

Recall: For a finite simplicial complex X, the chain complex of X is:

$$\cdots \xrightarrow{\delta_3} C_2(X) \xrightarrow{\delta_2} C_1(X) \xrightarrow{\delta_1} C_0(X) \xrightarrow{\delta_0} 0$$

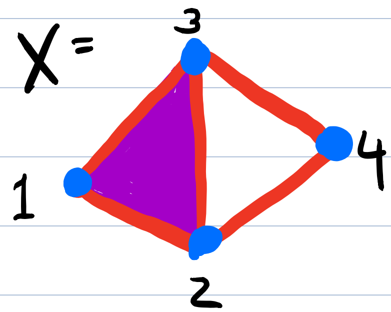
For $j \geq 0$, $Z_j(X) = \ker(\delta_j)$, $B_j(X) = \operatorname{im}(\delta_{j+1})$. **Prop:** $B_j(X) \subset Z_j(X)$.

Definition: For $j \geq 0$, the j^{th} homology vector space of X denoted $H_j(X)$, is the quotient space $Z_j(X)/B_j(X)$.

Intuitively, $\dim(H_j(X)) = \#$ j-dimsional holes of X.

$$\begin{array}{c} \dim(H_j(X)) \\ \shortparallel \\ \dim(Z_j(X)) - \dim(B_j(X)). \end{array}$$

Example: $X =$



We saw earlier that $\{\overbrace{[1,2]+[2,3]+[1,3]}^{b_1}, \overbrace{[2,3]+[2,4]+[3,4]}^{b_2}\}$ is a basis for $Z_1(X)$, so $\dim(Z_1(X)) = 2$.

We also saw that $\{b_1\}$ is a basis for $B_1(X)$, so $\dim(B_1(X)) = 1$.

Hence, $\dim(H_1(X)) = 2 - 1 = 1$, reflecting that X has a single 1-D hole.

Let us write $Z_1(X)$ as Z and $B_1(X)$ as B.

Definition of Quotient Spaces

As above, let V be any vector space and $W \subset V$ be a subspace.

We will illustrate the definitions with the example $V = Z$, $W = B$.

First, we define V/W as a set:

Recall that for V an abstract vector space and $v, v' \in V$, $v - v'$ is shorthand for $v + (-v')$.

Note: When $F = F_2$, $w + w = \vec{0}$, so $-w = w$, and $v - w = v + w$!

Define an equivalence relation $\sim$ on V by $v \sim v'$ if and only if $v - v' \in W$.

Let's check that this is really an equivalence relation:

Reflexivity: $\forall\, v \in V$, $v - v = \vec{0} \in W$ ✓

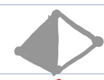
Symmetry: $v \sim w \Rightarrow v - w \in W \Rightarrow -(v - w) \in W$
$\Rightarrow w - v \in W$.

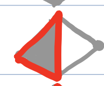
Transitivity: $v \sim w$ and $w \sim v \Rightarrow v - w \in W$ and $w - x \in W$.
Therefor $(v - w) + (w - x) = v - x \in W$, so $v \sim x$.

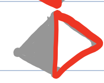
As a set, we define V/W to be $V/\sim$, the set of equivalence classes of $\sim$.

In the case where $V = Z_j(X)$, $W = B_j(X)$ for some X and $j \geq 0$, two cycles are equivalent if their difference is a boundary.

<u>Example</u> Let's determine Z/B as a set.

$Z = \{ \ \vec{0},$ 

$b_1 = [1,2] + [1,3] + [2,3],$ 

$b_2 = [2,3] + [3,4] + [2,4],$ 

$b_3 \ [1,2] + [2,4] + [3,4] + [1,3]. \ \}$ 

$B = \{ \vec{0}, b_1 \}.$

$b_1 - \vec{0} = b_1 \in B$, so $b_1 \sim \vec{0}$.

<u>Interpretation</u>: The cycle b_1 doesn't count as a hole, because it's a boundary.

$$\begin{aligned} b_2 - b_3 = b_2 + b_3 &= [2,3] + \cancel{[3,4]} + \cancel{[2,4]} + \\ &\quad [1,2] + \cancel{[2,4]} + \cancel{[3,4]} + [1,3] \\ &= [1,2] + [2,3] + [1,3] = b_1 \in W, \end{aligned}$$

So $b_2 \sim b_3$.

<u>Interpretation</u>: b_2 and b_4 represent the same hole, because geometrically, $[2,3]$ can be continuously deformed in X to $[1,2] \cup [1,3]$.

$b_1 \not\sim b_2$ because $b_1 - b_2 = b_1 + b_2 = b_3 \notin B$.

Thus as a set $Z/B = Z/\sim = \{\{\vec{0}, b_1\}, \{b_2, b_3\}\}$.

<u>Recall</u>: For any set S, $\sim$ an equivalence relation on S, and $x \in S$, $[x]$ denotes the equivalence class containing x.

<u>Example</u>: $[b_2] = [b_3] = \{b_2, b_3\}$

$$[0] = [b_1] = \{\vec{0}, b_1\} = B.$$

Equivalence classes are disjoint, so $\forall\ x, y \in S$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

Now, we define the vector space structure on V/W. We need to define addition and scalar multiplication.

We define $[v] + [w] = [v + w] \quad \forall\ v, w \in V$

$$c[v] = [cv] \quad \forall\ v \in V,\ c \in F.$$

$$\text{Then } \vec{0} \in V/W \text{ is } [\vec{0}] = W.$$

We need to check that $+$ and $\cdot$ do not depend on the choice of representative for the equivalence classes; otherwise, they are not well defined.

check that $+$ is well defined:

Suppose $[v]=[v']$. Then
$[w]=[w']$

Then $[v]+[w]=[v+w]$ and $[v']+[w']=[v'+w']$.

$$(v+w)-(v'+w') = \underbrace{v-v'}_{\text{in } W} + \underbrace{w-w'}_{\text{in } W} \in W$$

so $v+w \sim v'+w' \Rightarrow [v+w]=[v'+w']$.

This shows that addition is well defined in V/W.
The check that scalar multiplication is well defined is similar.

One can check that with these definitions V/W is indeed an abstract vector space over F.