AMAT 584 Lecture 27 4/3/2020

Today: Quotient Spaces and Homology, Continued
- Examples
- Induced maps on quotient spaces/homology.

Review

Let V be a vector space over $\mathbb{F}$, and $W < V$ a subspace.
We define V/W (also a vector space over $\mathbb{F}$), as follows:

Let $\sim$ be the equivalence relation on V given by
$v \sim v'$ iff $v - v' \in W$.

As a set, V/W is the set of equivalences classes of $\sim$.

Addition on V/W is defined by $[v]+[w]=[v+w]$
Scalar multiplication on V/W is defined by $c[v]=[cv]$.
Additive identity in V/W is $[\vec{0}]$ ($=[\vec{w}]$ for any $\vec{w} \in W$.)

Fact: $\forall\ v \in V$, $[v]=\{v+w \mid w \in W\}$.
Thus, $[v]$ is often denoted $v+W$.
Note that $\vec{0}+W=W$.
$v+W$ is called a coset.

this part is not review!

Homology For X a finite simplicial complex, we define
$H_j(X) = \underbrace{Z_j(X)}_{\text{cycles}} / \underbrace{B_j(X)}_{\text{boundaries}}$.

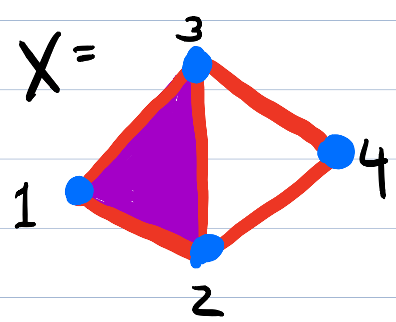
(End of review)

Proposition: Suppose V is finite dimensional, $W < V$ is a subspace with $\dim(W) = m$ and $\dim(V) = n$, and $\{v_1, \ldots, v_n\}$ is a basis for V such that $\{v_1, \ldots, v_m\}$ a basis for W. Then $\{[v_{m+1}], [v_{m+2}], \ldots, [v_n]\}$ is a basis for V/W. In particular, $\dim(V/W) = \dim(V) - \dim(W)$.

Phrased more colloquially: Extend a basis for W to a basis for V. The cosets of the elements in the extension form a basis for V/W.

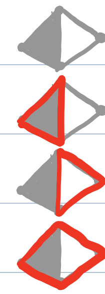
Let's revisit the example from last time:

Example: $X =$



As before, let's write $Z = Z_1(X)$
$B = B_1(X)$

$$Z = \{\vec{0},$$
$$z_1 = [1,2]+[2,3]+[1,3]$$
$$z_2 = [2,3]+[3,4]+[2,4]$$
$$z_3 = [1,2]+[2,4]+[3,4]+[1,3]\}.$$



$\{z_1, z_2\}$ is a basis for Z.

$B = \{\vec{0}, z_1\}$. $\{z_1\}$ is a basis for B.

We saw that as a set, $H_1(X) = Z/B = \{\{\vec{0}, z_1\}, \{z_2, z_3\}\}$
$$= \{B, z_2 + B\}.$$
$$= \{[\vec{0}], [z_2]\}$$

a couple of different ways of writing this.

additive identity.

According to the proposition, $\{[z_2]\}$ is a basis for $H_1(X)$ and this is easy to see directly. This makes good intuitive sense, as z_2 is a "hole."

It's easy to check that $H_1(X) \simeq \mathbb{F}_2$. isomorphism of vector spaces

Let's look at an example of addition in $H_1(X)$.

$$[z_1] + [z_2] = [z_1 + z_2] = [z_3] = [z_2]$$

calculation from previous lectures

In fact, this is the answer we expected, because $[z_1] = [\vec{0}] = B =$ additive identity in Z/B.

Let's look at $H_0(X)$ as well:

We explained in an earlier lecture that

$B_0(X)$ has basis $\Sigma_1 = \{[1]+[2], [2]+[3], [3]+[4]\}$
$Z_0(X) = C_0(X)$.

$C_0(X)$ has standard basis $\{[1],[2],[3],[4]\}$, but Σ is not a subset of this.

It can easily be checked, however, that

$\{[1], [1]+[2], [2]+[3], [3]+[4]\}$ is a basis for $Z_0(X)$.

Σ_1 is clearly a subset of this.

So by the proposition, $\{[1]+B_0\} = \{[[1]]\}$ is a basis for $H_0(X) = Z_0(X)/B_0(X)$.

Interpretation: $|X|$ has a single path component containing vertex 1.

Proposition: For any finite simplicial complex X, $\dim(H_0(X)) =$ # path components of $|X|$ = # components of 1-skeleton of X.

Moreover, if X has k components $X_1, .., X_k$ and for each $i \in \{1,..,k\}$, we choose a vertex $y_i \in X$, then $\{[y_1], [y_2], \dots, [y_k]\}$ is a basis for $H_0(X)$.

(But this is not the only for a basis for $H_0(X)$ can take.)

Induced maps on quotients

The following idea gives us the persistent part of persistent homology!

Proposition:
Let $f: V \to V'$ be a linear map, and let $W \subset V$, $W' \subset V'$ be subspaces such that $f(W) \subset W'$, (i.e., $f(w) \in W'$ for all $w \in W$).
Then f induces a linear map $f_*: V/W \to V/W$, given by
$f_*([v]) = [f(v)]$.

Pf: We need to check that f_* is well defined, i.e., if $[v] = [w]$ then $f_*([v]) = f_*([w])$.

If $[v] = [w]$ then $v \sim w$, i.e., $v - w \in W$.
$\Rightarrow f(v-w) = f(v) - f(w) \in W'$, $\Rightarrow f(v) \sim f(w)$
$\Rightarrow [f(v)] = [f(w)] \Rightarrow f_*([v]) = f_*([w])$. So f_* is well defined.

The linearity of f_* follows readily from the linearity of f.
I'll leave the details as an easy exercise. ∎

Corollary: A simplicial map $f: X \to Y$ induces a linear map $H_j(f): H_j(X) \to H_j(Y)$ for each $j \geq 0$.
(We sometimes denote $H_j(f)$ as f_*).

This requires some explanation, which we'll give next time.