AMAT 584 Lecture 28, 4/6/20

Today: Functoriality of homology

Homology is what's called a functor. [adjective form: functorial]

I won't define functors in general, because that would take too long. I'll just explain what this means for homology.

Functoriality of homology means that (for fixed $j \geq 0$)
1) for any simplicial complex X, we get a vector space $H_j(X)$
2) for any simplicial map $f: X \to Y$ we get a linear map

$$H_j(f): H_j(X) \to H_j(Y) \quad [H_j(f) \text{ is also denoted as } f_*]$$

such that $\underbrace{H_j(\mathrm{Id}_X) = \mathrm{Id}_{H_j(X)}}_{\text{homology takes maps to identity maps}}$ $\forall$ simplicial complexes X.

and $\underbrace{H_j(f \circ g) = H_j(f) \circ H_j(g)}_{\text{homology respects composition of maps}}$ $\forall$ $X \xrightarrow{f} Y \xrightarrow{g} Z$.

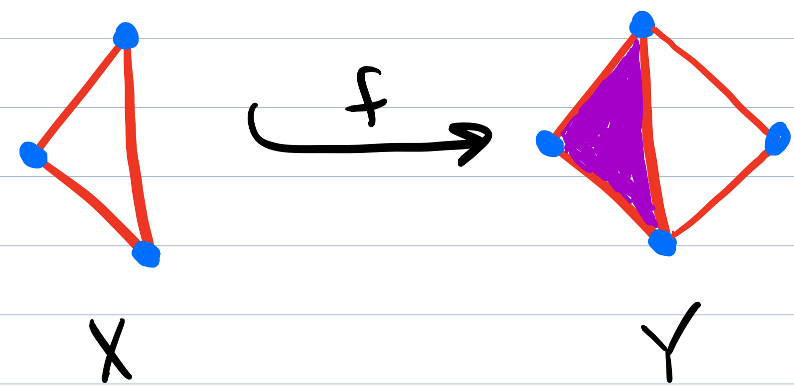
So far, we've focused only on 1), but 2) is the key to defining persistent homology

The induced maps $H_j(f)$ relate the holes in X and Y.

In this class, we'll primarily be interested in maps on homology induced by inclusions of simplicial complexes.

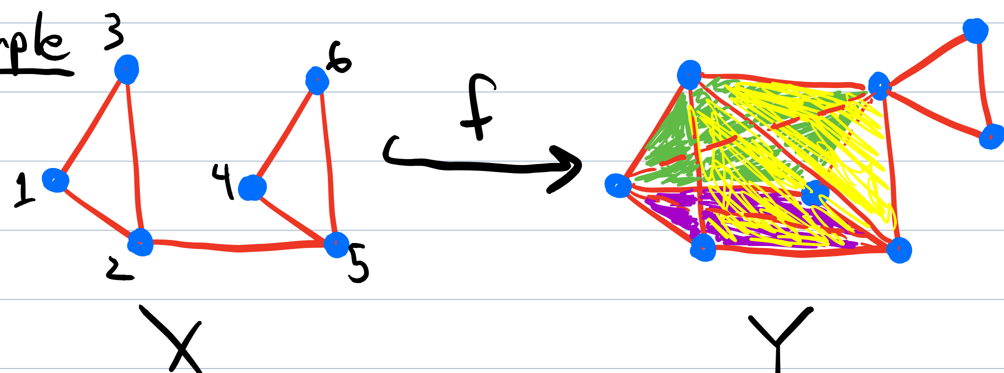
loose interpretation: For $f: X \hookrightarrow Y$ an inclusion of simplicial complexes, $\mathrm{rank}(H_j(f))$ is the number of j-D holes in X that remain holes in Y.

Example



$\dim(H_1(X)) = \dim(H_1(Y)) = 1$, but $\mathrm{rank}(H_1(f)) = 0$.
This expresses the fact that the hole in X closes up in Y.

Example



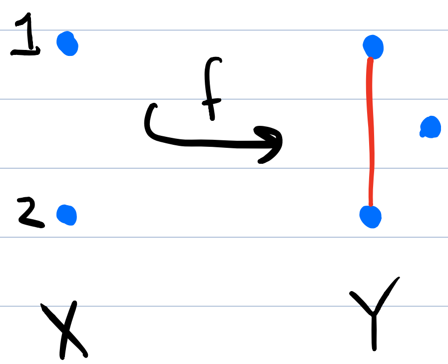
(triangle-shaped tube, with both end triangles not included, w/ another triangle glued on at a corner

$\dim(H_1(X)) = 2$
$\dim(H_1(Y)) = 2$
$\mathrm{rank}(H_1(f)) = 1$.

The cycles $[1,2]+[2,3]+[1,3]$ and
$[4,5]+[5,6]+[4,6]$
are not equivalent in $H_1(X)$, but become equivalent in $H_1(Y)$: Their sum is the boundary of
$t_1+t_2+\cdots t_6$, where $t_1,\ldots,t_6$ denote the 2-simplices of Y.

The fact that $\operatorname{rank}(H_1(f))=1$ expresses this "merging" of homology classes.

<u>Example</u>



$\dim(H_0(X))=2$
$\dim(H_0(Y))=2$
$\operatorname{rank}(H_0(f))=1.$ ⟵ This expresses the "merging" of the homology classes $[1]$ and $[2]$ in $H_0(Y)$.

<u>Definition of induced maps on homology</u>

A simplicial map $f: X \to Y$ induces a map
$$f_\#: C_j(X) \to C_j(Y)$$

First, we define $f_\#$ on $X^\sigma \in C_j(X)$, i.e. on chains with one term:

$$f_\#([x_0,\dots,x_j]) = \begin{cases} [f(x_0),\dots,f(x_j)] & \text{if } f(x_a)\neq f(x_b)\ \forall\ a\neq b \in \{0,\dots,j\} \\ 0 & \text{otherwise.} \end{cases}$$

Note that when f is an inclusion, the first condition always holds.

Then we define $f_\#$ on arbitrary chains in $C_j(X)$ by

$$f_\#(\sigma_1+\sigma_2+\cdots+\sigma_k) = f_\#(\sigma_1)+f(\sigma_2)+\cdots+f(\sigma_k).$$

<u>Proposition:</u> Each square in the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_3"] & C_2(X) \arrow[r, "\delta_2"] \arrow[d, "f_\#"] & C_1(X) \arrow[r, "\delta_1"] \arrow[d, "f_\#"] & C_0(X) \arrow[r, "\delta_0"] \arrow[d, "f_\#"] & 0 \arrow[d, "0"] \\
\cdots \arrow[r, "\delta_3"] & C_2(Y) \arrow[r, "\delta_2"] & C_1(Y) \arrow[r, "\delta_1"] & C_0(Y) \arrow[r, "\delta_0"] & 0
\end{tikzcd}$$

i.e., for each j, $\delta_j \circ f_\# = f_\# \circ \delta_j : C_j(X) \to C_{j-1}(Y)$

The proof is straightforward.

<u>Corollary:</u> (i) $f_\#(Z_j(X)) \subset Z_j(Y)$
(ii) $f_\#(B_j(X)) \subset B_j(Y)$

Proof: (i) If $z \in Z_j(X)$, then $\delta_j(z)=0$. $\delta_j \circ f_\#(z) = f_\# \circ \delta_j(\vec{z}) = f_\#(\vec{0}) = \vec{0}$. ∎

(The last equality uses the fact that $g(\vec{0})=\vec{0}$ for any linear map.)

(ii) If $z \in B_j(X)$ then $z = \delta_{j+1}(y)$ for some $y \in C_{j+1}(X)$.
$f_\#(z) = f_\#(\delta_{j+1}(y)) = \delta_{j+1}(f_\#(y))$, so $f_\#(z) \in B_j(Y)$. ∎

Thus $f_\#$ restricts to a map $f_\# : Z_j(X) \to Z_j(Y)$ such that $f_\#(B_j(X)) \subset B_j(Y)$.

Now recall the following general result about quotient spaces from last time:

Proposition:
Let $g: V \to V'$ be a linear map, and let $W \subset V$, $W' \subset V'$ be subspaces such that $f(W) \subset W'$, (i.e., $f(w) \in W'$ for all $w \in W$). Then g induces a linear map $g_*: V/W \to V/W$, given by $g_*([v]) = [g(v)]$.

Pf: We need to check that g_* is well defined, i.e., if $[v]=[w]$ then $g_*([v]) = g_*([w])$.

If $[v]=[w]$ then $v \sim w$, i.e., $v - w \in W$.
$\Rightarrow g(v-w) = g(v) - g(w) \in W'$, $\Rightarrow g(v) \sim g(w)$
$\Rightarrow [g(v)] = [g(w)] \Rightarrow g_*([v]) = g_*([w])$. So g_* is well defined.

The linearity of g_* follows readily from the linearity of g. I'll leave the details as an easy exercise. ∎

Applying the proposition with

$$V = Z_j(X) \qquad V' = Z_j(Y)$$
$$W = B_j(X) \qquad W' = B_j(Y)$$
$$g = f_\# : Z_j(X) \to Z_j(Y)$$

Gives us the induced map on homology
$H_j(f) : H_j(X) \to H_j(Y)$.

The check that this satisfies the functoriality conditions is straightforward. $\bullet$