AMAT 584 Lecture 29, April 8

<u>Today</u>: Finish Discussion of Induced Maps on Homology
Generalizations

<u>Review</u>

For any simplicial map $f: X \to Y$, with X and Y finite, we get

$H_j(f): H_j(X) \to H_j(Y)$ (also denoted f_*)

such that
$$\left.\begin{aligned} &1.\ H_j(Id_X) = Id_{H_j(X)} \\ &2.\ H_j(g \circ f) = H_j(g) \circ H_j(f) \end{aligned}\right\} \text{functoriality conditions.}$$

$H_j(f)$ is constructed in three steps.

1. Define linear maps $f_\#: C_j(X) \to f_\#(Y)$ $\forall j \geq 0$.

<u>Proposition</u>: These yield a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_3"] & C_2(X) \arrow[r, "\delta_2"] \arrow[d, "f_\#"] & C_1(X) \arrow[r, "\delta_1"] \arrow[d, "f_\#"] & C_0(X) \arrow[r, "\delta_0"] \arrow[d, "f_\#"] & 0 \\
\cdots \arrow[r, "\delta_3"] & C_2(Y) \arrow[r, "\delta_2"] & C_1(Y) \arrow[r, "\delta_1"] & C_0(Y) \arrow[r, "\delta_0"] & 0
\end{tikzcd}$$

2. From the commutativity of this diagram, we have:

<u>Corollary</u>: (i) $f_{\#}(Z_j(X)) \subset Z_j(Y)$,
(ii) $f_{\#}(B_j(X)) \subset B_j(Y)$.

The proof of this is in last lecture's notes, but we didn't cover it in class.

Proof: (i) If $z \in Z_j(X)$, then $\partial_j(z) = 0$. $\partial_j \circ f_{\#}(z) = f_{\#} \circ \partial_j(\vec{z}) = f_{\#}(\vec{0}) = \vec{0}$. ∎

(The last equality uses the fact that $g(\vec{0}) = \vec{0}$ for any linear map.)

(ii) If $z \in B_j(X)$ then $z = \partial_{j+1}(y)$ for some $y \in C_{j+1}(X)$.
$f_{\#}(z) = f_{\#}(\partial_{j+1}(y)) = \partial_{j+1}(f_{\#}(y))$, so $f_{\#}(z) \in B_j(Y)$. ∎

Thus $f_{\#}$ restricts to a map $f_{\#}: Z_j(X) \to Z_j(Y)$ such that $f_{\#}(B_j(X)) \subset B_j(Y)$.

<u>Note</u>: The only thing this proof uses about the maps $f_{\#}$ is that the diagram above commutes.

By the corollary, $f_{\#}$ restricts to a map $f_{\#}: Z_j(X) \to Z_j(Y)$ with $f_{\#}(B_j(X)) \subset B_j(Y)$.

3. Recall the following general fact about quotients:

<u>Proposition:</u>
Let $g: V \to V'$ be a linear map, and let $W \subset V$, $W' \subset V'$ be subspaces such that $g(W) \subset W'$.
Then f induces a linear map $g_*: V/W \to V/W$, given by $g_*([v]) = [f(v)]$.

We apply the proposition with
$$V = Z_j(X) \qquad V' = Z_j(Y)$$
$$W = B_j(X) \qquad W' = B_j(Y)$$
$$g = f_\#$$

To get the map $H_j(f): H_j(X) \to H_j(Y)$ such that for any $z \in Z_j(X)$,
$$H_j(f)([z]) = [f_\#(z)].$$

All of the above is review from the last lecture, just with some details filled in.

The proof that these induced maps satisfy the functoriality conditions is straightforward.

Generalizations

Homology of Infinite Simplicial Complexes:

So far, our treatment of homology has been for finite simplicial complexes.

Everything extends to infinite simplicial complexes, with just one change: Define $C_j(X)$ to be the set of all finite subsets of X^j. This is a subspace of the vector space $C_j(X)$.

Homology Over Fields Other Than $\mathbb{F}_2$

Our definition of homology yields a vector space over $\mathbb{F}_2$. The construction generalizes to yield a homology vector space over any field F (or even in an abelian group, like $\mathbb{Z}$)

This requires a couple of modifications to our approach. For simplicity, let me restrict attention to finite simplicial complexes.

Modification 1: Define $C_j(X) = \mathrm{Fun}(X^j, F)$.

(recall that we have shown that $P(S)$ is isomorphic to $\mathrm{Fun}(S, \mathbb{F}_2)$, so for finite simplicial complexes this indeed is a generalization of our definition, up to isomorphism).

Modification 2: When we define boundary maps, we need to put some negative signs in appropriate places to ensure that $\delta_{j-1} \circ \delta_j = 0$.

Why bother with homology over other fields?
Homology over F_3, for example, can detect subtle topological structure missed by homology over F_2 (and vice versa).

[See Hatcher's text for details]

Homology for Arbitary Topological Spaces

A variant of homology can be defined for arbitrary topological spaces. This is called Singular Homology

Rough idea (over F_2 coefficients):
For X a topological space, let X^j = Set of all continuous maps from a j-simplex into X. Note that X^j is a huge set!

Let $C_j(X)$ be the set of all finite subsets of X^j. Then X^j is a basis for $C_j(X)$, as in the simplicial case.

For $\sigma \in X^j$, define $\delta(\sigma)$ by summing the restrictions of σ to each of its $(j-1)$-dimensional faces.

Then define $\delta_j(\sigma_1 + \cdots + \sigma_k) = \delta(\sigma_1) + \cdots + \delta(\sigma_k)$, as in the simplicial case.

The rest of the def of homology works the same way.

For $f: X \to Y$ any continuous map, we also get an induced map on homology

), and let W $W' \subset$ $\subset$

$$H_j(f): H_j(X) \to H_j(Y).$$

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satisfying the same functoriality properties.

The definition of $H_j(f)$ is analogous to the simplicial case, though $f_\#$ must be defined slightly differently. Namely, for $\sigma \in X^j$, define $f_\#(\sigma) = f \circ \sigma$.
The rest of the definition is the same as in the simplicial case.

Theorem: If $f: X \to Y$ is a homotopy equivalence, $H_j(f): H_j(X) \to H_j(Y)$ is an isomorphism for all $j \geq 0$.

<u>Theorem</u>: For any simplicial complex X, $H_j(X) = H_j(|X|)$ for all $j \geq 0$.