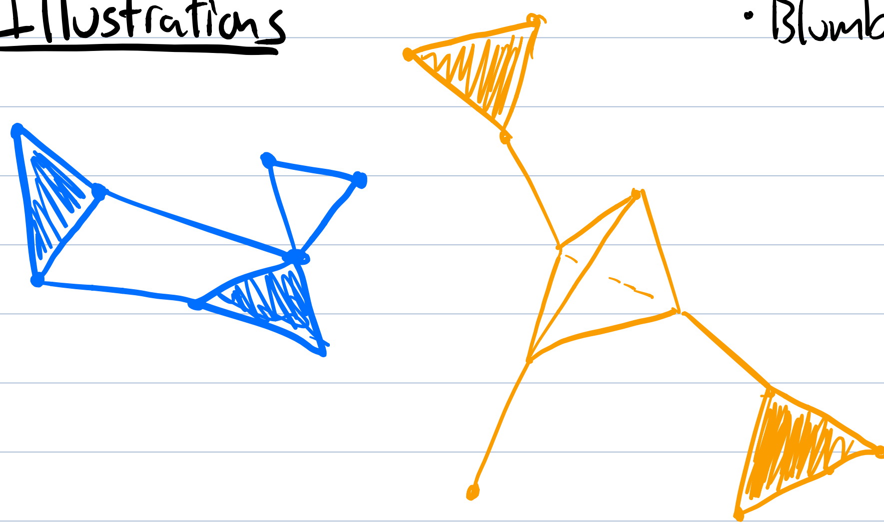
AMAT 584 Lecture 2, 1/24/20

Simplicial Complexes

references: · Edelsbrunner/Harer,
· Munkres, Elements of
· Algebraic Topology
· Blumberg/Rabadan

- Generalizations of graphs

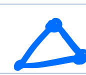
Illustrations

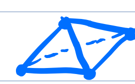


Simplices are certain simple subsets of $\mathbb{R}^n$.

A 0-simplex is a point

A 1-simplex is a line segment 

A 2-simplex is a triangle 

A 3-simplex is a solid tetrahedron 

To give the precise definition of a simplex, we will need a few preliminary notions.

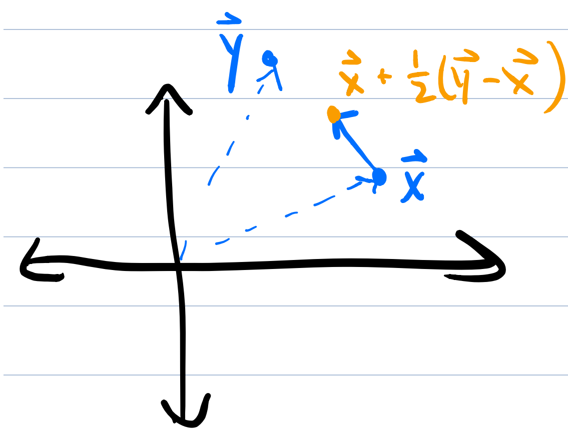
First, to motivate what follows, let's consider the following simple question:

<u>Question</u>: Given $\vec{x} \neq \vec{y} \in \mathbb{R}^n$, how do we describe the line segment L connecting $\vec{x}$ and $\vec{y}$ as a set?

<u>Answer</u>:

$$L = \{\vec{x} + c(\vec{y} - \vec{x}) \mid c \in [0,1]\}$$

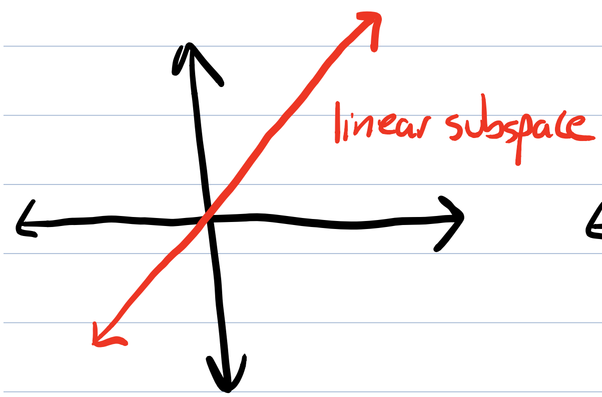
$$= \{c\vec{y} + (1-c)\vec{x} \mid c \in [0,1]\}.$$

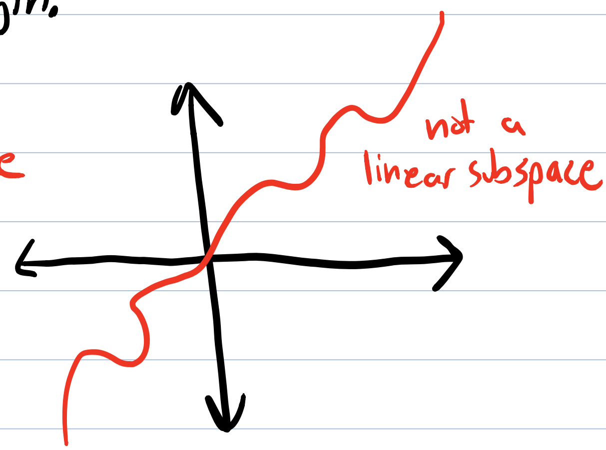


$$= \{c_1\vec{y} + c_2\vec{x} \mid c_i \geq 0,\ c_1 + c_2 = 1\}.$$

A subset $S \subset \mathbb{R}^n$ is called a <u>linear subspace</u> if it is the solution set to a linear equation $A\vec{x} = 0$ (A is a matrix.)

Geometrically, a linear subspace is flat (doesn't curve), and passes through the origin.

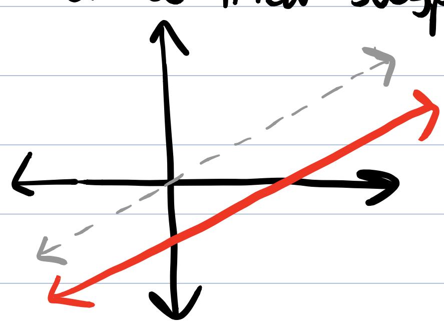




An <u>affine</u> subspace of $\mathbb{R}^n$ is the solution set to an equation $A\vec{x} = b$ (A is a matrix, b is a vector).

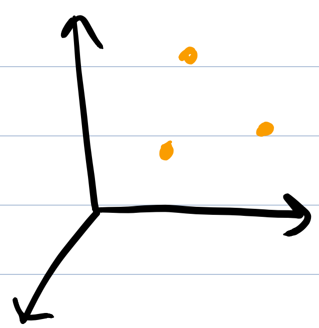
Geometrically, has the same shape as a linear subspace, but needn't pass through the origin.

That is, an affine subspace is a <u>translation</u> of a linear subspace.

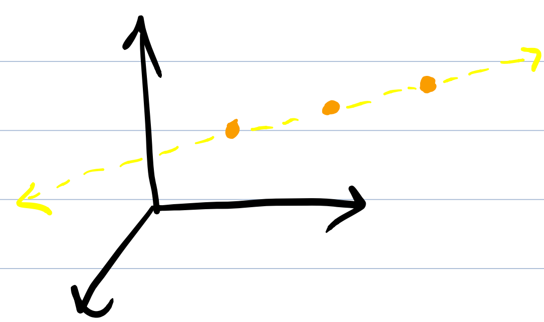


Def: A set X of k points in $\mathbb{R}^n$ is said to be in <u>general position</u> if X does not lie on any $(k-2)$-dimensional affine subspace.

For example, 3 points in $\mathbb{R}^3$ are in general position if they don't all line on a single line

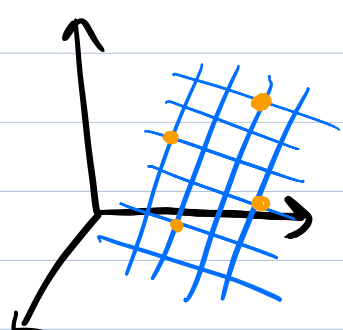


General Position



Not in general position

Four points in $\mathbb{R}^3$ are in general position if they don't lie on a plane.



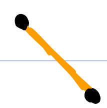
Not in general position

<u>Note</u>: If k points in $\mathbb{R}^n$ are in general position, then $n > k-2$.

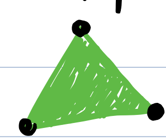
For $X = \{\vec{x}_1, \ldots, \vec{x}_n\} \subset \mathbb{R}^n$, the <u>convex</u> <u>hull</u> of X, is the set

$$\mathrm{Conv}(X) = \{ c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \cdots + c_n\vec{x}_n \mid \text{each } c_i \geq 0 \text{ and } \sum_{i=1}^{n} c_i = 1 \}.$$

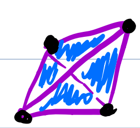
- The convex hull of a single point is the singleton set containg only that point.
- The convex hull of two points is the line segment connecting them.



- The convex hull of three points in general position is the triangle with these points as vertices.

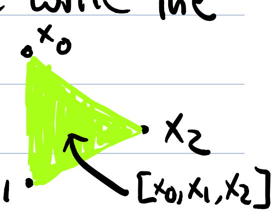


- The convex hull of four points in general position is a solid tetrahedron with these points as vertices.



<u>Definition</u>: A (k-)<u>simplex</u> is the convex hull of a set of $k+1$ points in general position.

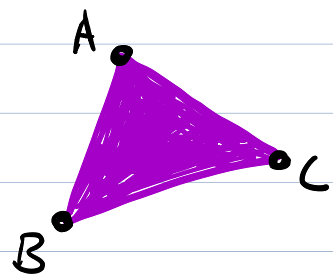
If $x_0, \ldots, x_k$ are in general position, we write the associated simplex as $[x_0, \ldots, x_k]$.



I'll sometimes call this the simplex spanned by $\{x_0, \ldots, x_k\}$

Definition: A face of a simplex $[x_0, \ldots x_k]$ is the simplex spanned by a non-empty subset of $\{x_1, \ldots, x_k\}$.

Ex:

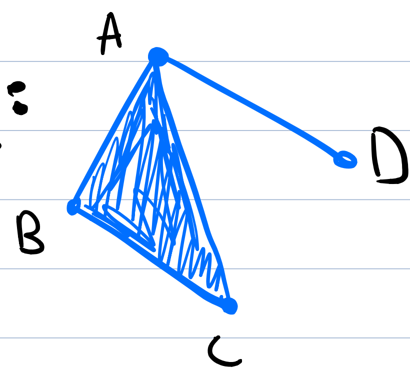


The faces of $[A,B,C]$ are:
$[A]$, $[B]$, $[C]$,
$[A,B]$, $[B,C]$, $[A,C]$,
$[A,B,C]$.

Definition: A (geometric) simplical complex is a set S of simplices in $\mathbb{R}^n$ (for some fixed n) such that

1. each face of a simplex in S is contained in S
2. the intersection of two simplices in X is a face of each of them (if non empty).

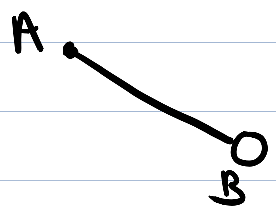
Example:



Let $A, B, C, D \in \mathbb{R}^2$ be as shown

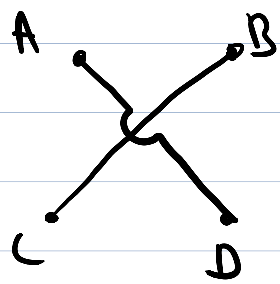
This illustrates the simplicial complex
$\{[A], [B], [C], [D], [A,B], [A,C], [B,C], [A,D], [A,B,C]\}$.

Example: For $A, B \in \mathbb{R}^2$ as shown,



$\{[A],[AB]\}$ is not a simplical complex: property 1 is violated.

Example: For $A, B, C, D \in \mathbb{R}^2$ as shown,



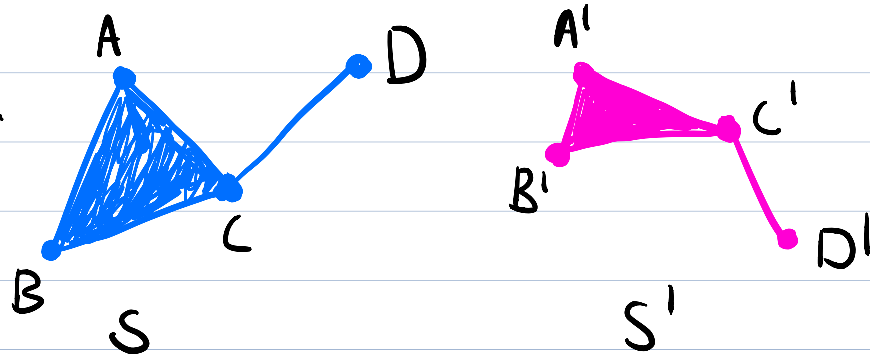
$\{[A],[B],[C],[D],[A,D],[B,C]\}$ is not a simplial complex: Property 2. is violated.

Definition: For S a simplicial complex, we call the union of the simplices in S the geometric realization of S, and denote this $|S|$.

Abstract Simplicial Complexes

Motivation: It turns out that up to homeomorphism, $|S|$ doesn't depend on the position of the 0-simplices of S.

Example:



$|S| \cong |S'|$. (Recall that $\cong$ means "is homeomorphic to").

Let's make this precise:

Proposition: Let S and S' be simplicial complexes, and suppose there is a bijection f from the 0-simplices of S to the 0-simplices of S' such that

$$[x_0, \dots, x_k] \in S \text{ iff } [f(x_0), \dots, f(x_k)] \in S'.$$

Then $|S| \cong |S'|$.

Def: an abstract simplicial complex on a set S is a collection X of non-empty finite subsets such that if $\sigma \in X$ and $\emptyset \neq \tau \subset \sigma$, then $\tau \in S$.

The subsets of size $(k+1)$ are called k-simplices of X.

The simplex $\{a_0, .., a_k\}$ is written $[a_0, .., a_k]$.