Today: Some computations

Example: Let $S = \{a, b, c, d\}$

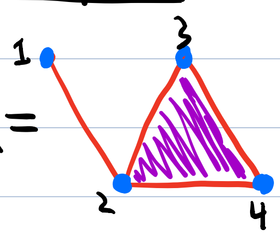
The power set $P(S)$ is a vector space with ordered basis S.
(That is, $\{\{a\}, \{b\}, \{c\}, \{d\}\}$ is a basis for $P(S)$, but we identify this subset of $P(S)$ with S.)

What is $[\{b,c\}]_S \in F_2^4$, the column vector representation of $\{b,c\}$ with respect to this ordered basis?

$$\{b,c\} = \{b\} \cup \{c\} = \{b\} + \{c\}$$
$$= b + c$$
$$= 0a + 1b + 1c + 0d.$$

So $[\{b,c\}]_S = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

Example:

$X =$ 

Problem: a. Represent each non-zero map $\delta_j : C_j(X) \to C_{j-1}(X)$ as a matrix with respect to the standard bases for

$C_j(X)$ and $C_{j-1}(X)$. Recall, the standard basis for $C_i(X)$ is X^i.

b. Find bases for each $Z_j(X)$, $B_j(X)$.

Key tool: Gaussian elimination.

Solution:

a. $X^0 = \{[1],[2],[3],[4]\}$ $\quad X^2 = \{[2,3,4]\}$
$X^1 = \{[1,2],[2,3],[2,4],[3,4]\}$.

Let us denote the matrix representation of δ_j, with respect to the given orderings of the X^j, by $[\delta_j]$.
[see lecture 21].

Let σ_i^j denote the i^{th} entry of X^j.

By definition, $[\delta_j]$ is the matrix whose i^{th} column is $[\delta_j(\sigma_i^j)]_{X^{j-1}}$

Remember, for $\sigma \in X^j$,
$\delta_j(\sigma) = \delta(\sigma) =$ sum of all $(j-1)$-dimensional faces of σ.

$$[\delta_1] = \left([\delta([1,2])]_{X^0} \,\middle|\, [\delta([2,3])]_{X^0} \,\middle|\, [\delta([2,4])]_{X^0} \,\middle|\, [\delta([3,4])]_{X^0} \right)$$

$$= \left(\left[[1]+[2]\right]_{X^0} \,\middle|\, \left[[2]+[3]\right]_{X^0} \,\middle|\, \left[[2]+[4]\right]_{X^0} \,\middle|\, \left[[3]+[4]\right]_{X^0} \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$[\delta_2] = \left([\delta([2,3,4])]_{X^1} \right)$$

$$= \left(\left[[2,3]+[2,4]+[3,4]\right]_{X^1} \right)$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

$\delta_j = 0$ for $j \geq 2$, so we are done with a.

b. $Z_0(X) = C_0(X)$, so X^0 is a basis for $Z_0(X)$.

To find a basis for $B_j(X)$, we do Gaussian elimination on the columns of $[\delta_{j+1}]$. The non-zero columns of the resulting matrix represent a basis for $B_j(X)$.

Brief justification:

Proposition: For any finite dimensional vector space V over F and ordered basis B for V the function $f: V \to F^{|B|}$ is an isomophism.

Since f is an iso, it preserves all the algebraic structure of V. This means that to find a basis for a subspace of V, we can find a basis for the corresponding subspce of $F^{|B|}$, and then map the elements back into V via f^{-1}.

Column-wise Gaussian elimination on $[\delta_1]$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{add col 2 to col 3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{add col. 3 to col 4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The non-zero columns are the first 3. They represent the basis $\{[1]+[2], [2]+[3], [3]+[4]\}$ for $B_0(X)$.

$$[\delta_2] = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since it has 1 column, the matrix is already column-reduced.

This column represents $[2,3]+[2,4]+[3,4] \in C_1(X)$.

Thus, $\{[2,3]+[2,4]+[3,4]\}$ is a basis for $B_1(X)$.

To compute a basis for $Z_j(X) = \ker(\delta_j)$, we find a basis for the null space of $[\delta_j]$ by solving the linear system $[\delta_j]\vec{x} = \vec{0}$ for $\vec{x}$.

[The justification for this is similar to the justification for our approach to computing a basis for $B_j(X)$.]

To solve the linear system, we do row-wise Gaussian elimination on $[\delta_j]$.

Now let's consider the case $j=1$:

$$[\delta_1] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{add row 1 to row 2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{add row 2 to row 3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{add row 3 to row 4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now use back substitution:

$x_1 = 0$

$x_2 + x_3 = 0$

$x_3 + x_4 = 0$

x_4 is a free variable

taking $x_4 = 1$, we find that $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is the unique non-zero solution

so $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for the null space of $[\delta_1]$.

This represents $[2,3]+[2,4]+[3,4] \in Z_1(X)$,
so

$\{[2,3]+[2,4]+[3,4]\}$ is a basis for $Z_1(X)$.