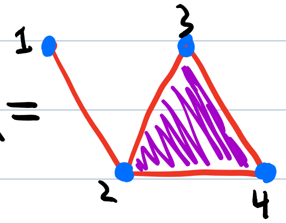
AMAT 584 Lecture 31, 4/13/20

<u>Today</u>: Finish the example from last lecture

Recall the example we were considering last lecture:

$X=$ 

We found that with respect to the standard bases for the vector spaces $C_j(X)$, ordered as follows,

$X^0=\{[1],[2],[3],[4]\}$
$X^1=\{[1,2],[2,3],[2,4],[3,4]\}$,

$$\delta_1: C_1(X)\to C_0(X) \qquad \delta_2: C_2(X)\to C_1(X)$$

are represented by the matrices

$$[\delta_1]=\begin{pmatrix}1&0&0&0\\1&1&1&0\\0&1&0&1\\0&0&1&1\end{pmatrix}, \quad [\delta_2]=\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

By performing Gaussian elimination on the columns of $[\partial_1]$, we found that

$$\{[1]+[2], [2]+[3], [3]+[4]\} \text{ is a basis for } B_0(X).$$

Clearly $\{[2,3]+[2,4]+[3,4]\}$ is a basis for $B_1(X)$.

$B_j(X) = 0$ for $j \geq 2$ because $\partial_{j+1} = 0$ for $j \geq 2$.

Now we compute bases for each $Z_j(X)$.

$Z_0(X) = C_0(X)$, so X^0 is a basis for $Z_0(X)$.

To compute a basis for $Z_j(X)$, $j \geq 1$, is more involved

Recall: The null space of an $m \times n$ matrix A with coefficients in a field F is the subspace

$$\operatorname{null}(A) = \{\vec{x} \in F^n \mid A\vec{x} = \vec{0}\}.$$

To find a basis for $\operatorname{null}(A)$ we solve the linear system $A\vec{x} = \vec{0}$ using Gaussian elimination (on rows).

[It is also possible to solve the system using Gaussian elimination on columns, but we will not discuss this now.]

To compute a basis for $Z_j(X) = \ker(\delta_j)$, we find a basis for the null space of $[\delta_j]$.

<u>Fact</u>: The elements of this basis represent the elements of a basis for $Z_j(X)$.

Now let's consider the case $j=1$:

$$[\delta_1] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{add row 1 to row 2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{add row 2 to row 3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{add row 3 to row 4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now use back substitution to find a basis for $Null([\delta_1])$:

$x_1 = 0$

$x_2 + x_3 = 0$

$x_3 + x_4 = 0$

x_4 is a free variable

The set of solutions to this system is:

$$\left\{ x_4 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \;\middle|\; x_4 \in F_2 \right\},$$ So $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $Null([\delta_1])$.

The column vector $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ represents $[2,3]+[2,4]+[3,4] \in Z_1(X)$ with respect to the standard basis for $C_1(X)$, so

$\{[2,3]+[2,4]+[3,4]\}$ is a basis for $Z_1(X)$.

Now let's compute a basis for $Z_2(X)$.

$$[\delta_2] = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

$Null([\delta_2]) = \{\vec{0}\} \in F_2^1$, so $Null([\delta_2])$ has an empty basis

$\Rightarrow Z_2(X)$ has an empty basis

$\Rightarrow Z_2(\vec{X}) = \vec{0}$.

Since $C_j(X) = \{\vec{0}\}$ for $j \geq 3$, $Z_j(X) = \{\vec{0}\}$ for $j \geq 3$ as well.

<u>Exercise</u>: Compute a basis for each homology vector space of X.

The key tool is this proposition from lecture 27:

<u>Proposition</u>: Suppose V is finite dimensional, $W < V$ is a subspace with $\dim(W) = m$ and $\dim(V) = n$, and $\{v_1, \ldots, v_n\}$ is a basis for V such that $\{v_1, \ldots, v_m\}$ a basis for W. Then $\{[v_{m+1}], [v_{m+2}], \ldots, [v_n]\}$ is a basis for V/W.

Using this, we see that $H_1(X)$ has the empty basis, i.e., $H_1(X)$ is trivial.

To compute a basis for $H_0(X)$, we need to first extend the basis we computed for $B_0(X)$ to one for $Z_0(X)$. This can be done by doing Gaussian elimination on the columns of the matrix:

column representation of our basis for B_0

$$\left(\begin{array}{ccc|cccc} 1 & & & 1 & 0 & 0 & 0 \\ 1 & 1 & & 0 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 1 & 0 \\ & & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{cols}]{\text{G.E. on}} \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

column representation of our basis for Z_0.

After reduction, the non-zero columns on the right give an extension of our basis for B_0 to one for Z_0.