AMAT 584 Lecture 32, 4/15/20

<u>Today</u>: Singular Homology, continued
Persistence Modules.

For our discussion of singular homology, it will be useful to introduce the following abstraction of our construction of simplicial homology:

<u>Def</u>:
A <u>chain complex</u> C is a sequence of vector spaces and linear maps

$$\cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} 0$$

such that $\delta_{j-1} \circ \delta_j = 0$ for all $j \geq 0$.

For any chain complex, we can define the j^{th} homology module of C $\forall\, j \geq 0$, denoted $H_j(C)$, in exactly the same way we did for simplicial homology.

<u>Def</u>:
A <u>chain map</u> $f: C \to D$ between chain complexes C and D is a choice of linear maps $f_j: C_j \to D_j$ for each $j \geq 0$, such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[r, "\delta_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\delta_0"] \arrow[d, "f_0"] & 0 \\
\cdots \arrow[r, "\delta_3"] & D_2 \arrow[r, "\delta_2"] & D_1 \arrow[r, "\delta_1"] & D_0 \arrow[r, "\delta_0"] & 0
\end{tikzcd}$$

For any chain map $f: C \to D$, we can define an induced map on homology

$$H_j(f): H_j(C) \to H_j(D)$$

for all $j \geq 0$, in exactly the same way we defined the induced maps on singular homology.

<u>Singular</u> <u>Homology</u>

<u>Recall</u>: This is a construction of homology for arbitrary topological spaces.

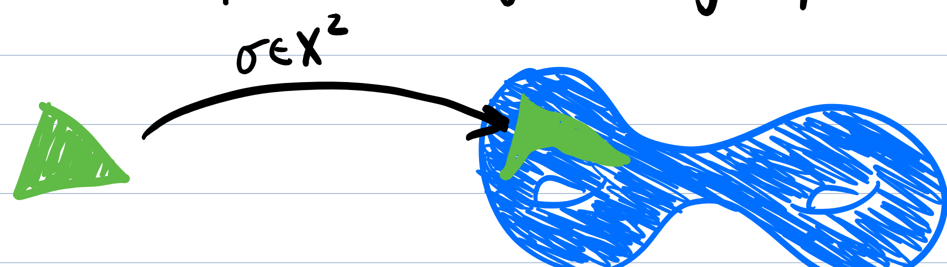
To define it, we:

- construct a chain complex $C(X)$ for each topological space
- construct a chain map $f_\#: C(X) \to C(Y)$ for each continuous map $f: X \to Y$.

Then, by the abstract discussion above, we get homology vector spaces $H_j(X)$ $\forall j \geq 0$ and induced maps $H_j(F): H_j(X) \to H_j(Y)$
The induced maps satisfy the same functoriality properties as the induced maps in singular homology.

<u>Definition of the singular chain complexes</u> (review)
We give the construction over the field $\mathbb{F}_2$, though it extends to arbitrary fields.

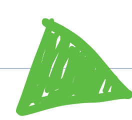
For X a topological space, let X^j denote the set of all continuous maps from a (geometric) j-simplex into X.

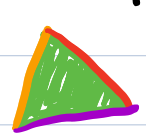


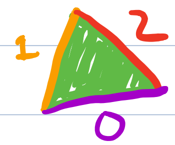
Elements of X^j are called <u>singular j-simplices</u>
Note: X^j is usually a very large (infite set).

Let $C_j(X)$ denote the set of all finite subsets of X^j.

Note that a j-simplex has $j+1$ $(j-1)$-dimensional faces.

For example, the 2-simplex  has 3 1-dim faces:

 . As a notational convenience, let's assume these faces are labeled $0, .., j$



For $\sigma \in X^j$, and $i \in \{0, \dots, j\}$, let $\sigma|_i$ denote the restriction of σ to the i^{th} face of the j-simplex

Define $\delta(\sigma) = \sigma|_0 + \sigma|_1 + \cdots + \sigma|_j$.

Define $\delta_j : C_j(X) \to C_{j-1}(X)$ by

$$\delta_j(\sigma_1 + \sigma_2 + \cdots + \sigma_k) = \delta(\sigma_1) + \cdots + \delta(\sigma_n).$$

This completes the construction of the chain complex $C(X)$.

The argument that $\delta_{j-1} \circ \delta_j = 0$ is similar to the one in the simplicial case.

Given a continuous map $f: X \to Y$, the definition of the singular chain map

$f_\#: C(X) \to C(Y)$ is simple:

Define $(f_\#)_j: C_j(X) \to C_j(Y)$ by

$(f_\#)_j(\sigma_1 + \cdots + \sigma_k) = f\circ\sigma_1 + f\circ\sigma_2 + \cdots + f\circ\sigma_k.$

It is easy to check that this really defines a chain map. (i.e. that the ladder-shaped diagram in question really commutes).

Key theorems about singular homology

Theorem: If $f: X \to Y$ is a homotopy equivalence, $H_j(f): H_j(X) \to H_j(Y)$ is an isomorphism for all $j \geq 0$.

Theorem: For any simplicial complex X, $H_j(X) \cong H_j(\underbrace{|X|}_{\text{geometric realization of } X})$ for all $j \geq 0$.

Corollary: If two simplicial complexes X and Y have homeomorphic geometric realizations, then $H_j(X) \cong H_j(Y)$ for all $j \geq 0$.

Remark: Whereas simplicial homology is easy to compute via linear algebra, singular homology is less amenable to "naive" matrix computations, because the vector spaces of the singular chain complex are usually infinite-dimensional. There are well-developed algebraic tools for singular homology computations (e.g. "long exact sequences, Mayer-Vietoris sequences), but we will not consider these in this course.