AMAT 584 Lecture 34 4/20/20

Today: Persistent Homology

Note: Lec 33 was canceled. But I am assuming you have read the notes from that lecture. I will cover some of the same material but with different emphasis.

Persistent Homology Pipeline

Data (e.g. points in $\mathbb{R}^n$ or finite metric space)

↓

Filtration (e.g. Čech, Rips, or Delaunay filtration)

↓

Persistence Module

↓

Barcode

We explained earlier in the course how to construct a filtration from a persistence module. Now we need to consider the other two steps in this pipeline.

In the lecture 33 notes, we considered filtrations and persistence modules indexed by $[0,\infty)$ or $\mathbb{N}$.

Today, we will focus just on $\mathbb{N}$-indexed filtrations and persistence modules.

Definitions ($\mathbb{N}$-indexed case)

A filtration G is a sequence of simplicial complexes (or topological spaces)

$G_0 \subset G_1 \subset G_2 \subset \cdots$

(For simplicity, in what follows, we work with vector spaces over the field $\mathbb{F}_2$, though we can work with any field.)

A persistence module M is a sequence of vector spaces and linear maps.

$M_0 \xrightarrow{M_{0,1}} M_1 \xrightarrow{M_{1,2}} M_2 \xrightarrow{M_{2,3}} \cdots$.

$\boxed{\text{We say } M \text{ is p.f.d. if } \dim(M_i) < \infty \ \forall i.}$

A barcode is a multiset of intervals in the real line. Typically each interval is of the form $[a,b)$. b is allowed to be ∞.

$\boxed{\text{Filtration}}$

$\downarrow$

$\boxed{\text{Persistence Module}}$

Applying i^{th} homology to each simplicial complex and each inclusion map in a filtration

$$G = G_0 \xrightarrow{j_0} G_1 \xrightarrow{j_1} G_2 \xrightarrow{j_2} \cdots$$

Gives a persistence module

$$H_i G = H_i(G_0) \xrightarrow{H_i(j_0)} H_i(G_1) \xrightarrow{H_i(j_1)} H_i(G_2) \xrightarrow{H_i(j_2)} \cdots$$

$\boxed{\text{Persistence Module}}$

$\downarrow$

$\boxed{\text{Barcode}}$

Def: A compatible set of bases $\mathcal{B}$ for a persistence module M Is a choice of basis B_r for each vector space M_r of M, such that

1) if $\forall r \in \mathbb{N}$ and $b \in B_r$, either $M_{r,r+1}(b) \in B_{r+1}$ or $M_{r,r+1}(b) = 0$.
2) If $b_1, b_2 \in B_r$, $b_1 \neq b_2$, and $M_{r,r+1}(b_1) \neq 0$, then $M_{r,r+1}(b_1) \neq M_{r,r+1}(b_2)$. Think of this loosely as an injectivity property.

Example: $M = \underset{\overset{\shortparallel}{M_0}}{\mathbb{F}_2} \underset{\overset{\shortparallel}{M_{0,1}}}{\xrightarrow{\binom{1}{0}}} \underset{\overset{\shortparallel}{M_1}}{\mathbb{F}_2^2} \underset{\overset{\shortparallel}{M_{1,2}}}{\xrightarrow{(1\,0)}} \underset{\overset{\shortparallel}{M_2}}{\mathbb{F}_2} \underset{\overset{\shortparallel}{M_{2,3}}}{\rightarrow} 0 \rightarrow 0 \rightarrow \cdots$

let $B_0 = \{1\}$
$B_1 = \{\binom{1}{0}, \binom{0}{1}\}$
$B_2 = \{1\}$
$B_i = \{\}$ for $i \geq 3$.

$M_{0,1}(1) = \binom{1}{0}$ ✓
$M_{1,2}\binom{1}{0} = 1$ ✓
$M_{1,2}\binom{0}{1} = 0$ ✓
$M_{2,3}(1) = 0$

This gives a compatible basis $\mathcal{B}$.

__Example:__

$$M = \underset{M_0}{\mathbb{F}_2} \xrightarrow[]{M_{0,1} = \binom{1}{0}} \underset{M_1}{\mathbb{F}_2^2} \xrightarrow{M_{1,2} = (1\ 1)} \underset{M_2}{\mathbb{F}_2} \xrightarrow{M_{2,3}} 0 \to 0 \to \cdots$$

B as above is not a compatible basis because $M_{1,2}\binom{1}{0} = M_{1,2}\binom{0}{1} = 1$, so 2) is violated.

But the following is a compatible basis:

$B_0 = \{1\}$ | $M_{0,1}(1) = \binom{1}{0}$ ✓
$B_1 = \{\binom{1}{0}, \binom{1}{1}\}$ | $M_{1,2}\binom{1}{0} = 1$ ✓
$B_2 = \{1\}$ | $M_{1,2}\binom{1}{1} = 0$ ✓
$B_i = \{\}$ for $i \geq 3$. | $M_{2,3}(1) = 0$

__Constructing a barcode from a compatible basis.__

__Idea:__ Elements in a compatible basis form chains. The indices at which a chain begins and ends gives an interval in the barcode.

Suppose B is a compatible basis for a persistence module M.

Consider the set $\sqcup B = \{(b,r) \mid b \in B_r\}$. Let $\rho : \sqcup B \to [0,\infty)$ be given by $\rho(b,r) = r$. That is, ρ is projection onto the second coordinate.

Define an equivalence relation $\sim$ on $\sqcup \mathcal{B}$ by $(b,r) \sim (b',r')$ iff $M_{r,r'}(b) = b'$ or $M_{r',r}(b') = b$.

(It's straightforward to check that this is an equivalence relation.)

For each equivalence class E, let

$$b(E) = \min \rho(E), \qquad d(E) = \max \rho(E) + 1.$$

We define the barcode

$$\mathrm{Barc}(\mathcal{B}) = \{ [b(E), d(E)) \mid E \text{ is an eq. class of } \sim. \}$$

<u>Theorem</u>: If M is p.f.d. then there exists a compatible basis $\mathcal{B}$ for M. Moreover, $\mathrm{Barc}(\mathcal{B})$ is independent of the choice of $\mathcal{B}$. Thus we obtain a well-defined barcode $\mathrm{Barc}(M)$.