AMAT 584 Lecture 34, April 22

<u>Today</u>: More on constructing barcodes from persistence modules.
Intuitive interpretation of persistent homology.
Examples

Review of the last step of the persistence pipeline:

$\boxed{\text{Persistence Module}}$
$\downarrow$
$\boxed{\text{Barcode}}$

As with last lecture, we'll work in the $\mathbb{N}$-indexed case.

<u>Def</u>: A <u>compatible set of bases</u> $\mathcal{B}$ for a persistence module M is a choice of basis B_r for each vector space M_r of M, such that

1) if $\forall r \in \mathbb{N}$ and $b \in B_r$, either $M_{r,r+1}(b) \in B_{r+1}$ or $M_{r,r+1}(b) = \vec{0}$,
2) If $b_1, b_2 \in B_r$, $b_1 \neq b_2$, and $M_{r,r+1}(b_1) \neq 0$, then $M_{r,r+1}(b_1) \neq M_{r,r+1}(b_2)$. Think of this loosely as an injectivity property.

Define an equivalence relation $\sim$ on $\sqcup \mathcal{B} = \{(b,r) \mid r \in \mathbb{N}, b \in B_r\}$.
by $(b,r) \sim (b',r')$ if and only if $M_{r,r'}(b) = b'$ or
$M_{r',r}(b') = b$.

Here $M_{r,r'}$ denotes a composition of maps in the persistence module, similarly for $M_{r',r}$.

For each equivalence class E, let

$$b(E) = \min \rho(E), \qquad d(E) = \max \rho(E) + 1.$$

We define the barcode

$$\mathrm{Barc}(\mathcal{B}) = \{ [b(E), d(E)) \mid E \text{ is an eq. class of } \sim. \}$$

<u>Example</u>: Last time we considered the persistence module

$$M = \underset{\underset{M_0}{\parallel}}{\mathbb{F}_2} \xrightarrow{\binom{1}{0}} \underset{\underset{M_1}{\parallel}}{\mathbb{F}_2^2} \xrightarrow{(1\ 1)} \underset{\underset{M_2}{\parallel}}{\mathbb{F}_2} \to 0 \to 0 \to \cdots$$

And saw that the following is a compatible basis $\mathcal{B}$ for M:

$$B_0 = \{1\},\ B_1 = \{\tbinom{1}{0}, \tbinom{0}{1}\},\ B_2 = \{1\},\ B_i = \{\}\text{ for } i \geq 3.$$

$$\sqcup \mathcal{B} = \{(1,0), (\tbinom{1}{0}, 1), (\tbinom{0}{1}, 1), (1,2)\}.$$

$M_{0,1}(1) = \binom{1}{0}$, $M_{1,2}\binom{1}{0} = 1$, $M_{1,2}\binom{0}{1} = 0$, so

$\sqcup\mathcal{B}/\sim = \{E_1, E_2\}$, where $E_1 = \{(1,0), (\binom{1}{0},1), (1,2)\}$, $E_2 = \{\binom{0}{1}, 1\}$.

To compute $\rho(E_1)$ and $\rho(E_2)$, just take the second element in each pair:

$$(E_1) = \{0, 1, 2\} \qquad \rho(E_2) = 1.$$

$$b(E_1) = \min \rho(E_1) = 0 \qquad b(E_2) = \min \rho(E_2) = 1$$
$$d(E_1) = \text{make } \rho(E_1) + 1 = 3. \qquad d(E_2) = \max \rho(E_2) + 1 = 2.$$

So $\mathrm{Barc}(B) = \{[0,3), [1,2)\}$.

Recall the following:

<u>Theorem</u>: If M is p.f.d. then there exists a compatible basis B for M. Moreover, $\mathrm{Barc}(B)$ is independent of the choice of B. Thus we obtain a well-defined barcode $\mathrm{Barc}(M)$.

So in the example above, $\mathrm{Barc}(M) = \mathrm{Barc}(B) = \{[0,3), [1,2)\}$.

<u>Remarks about the theorem</u>:

- This theorem is called the <u>structure theorem for persistence modules</u>
- As indicated earlier, it extends to persistence modules indexed by $\mathbb{Z}$ or $[0,\infty)$, or $\mathbb{R}$.
- It is a variant of a standard theorem in linear algebra, the Jordan normal form theorem.

- The proof of the theorem is not too painful, but lies well beyond the scope of this course.

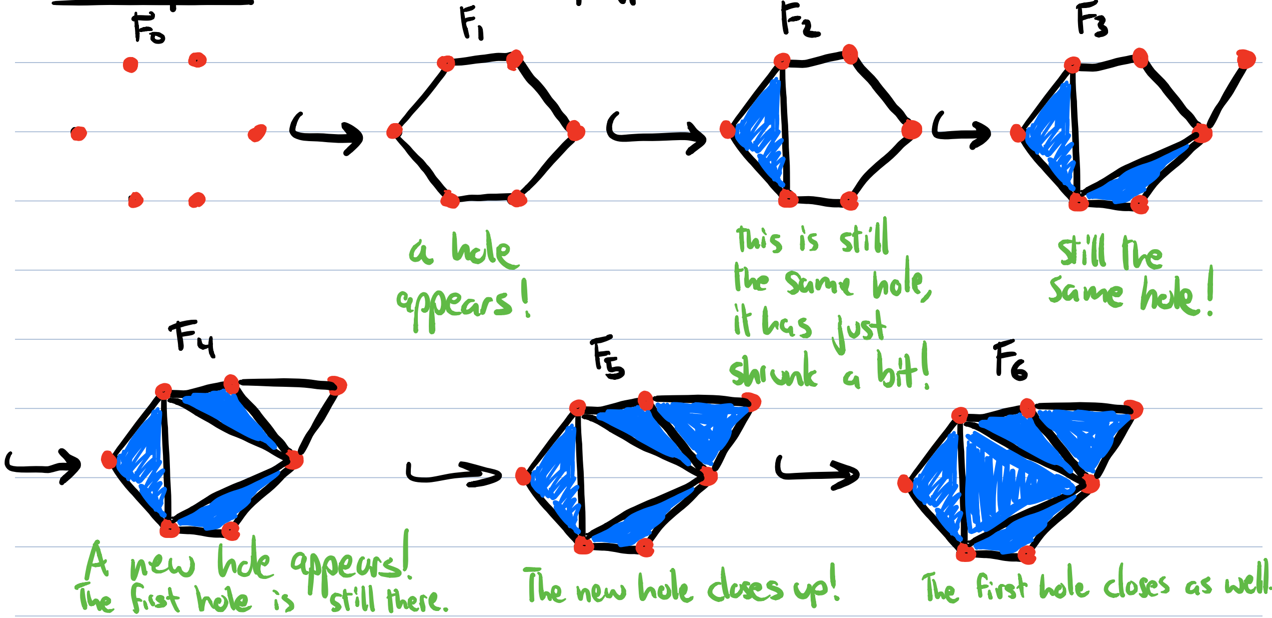
So now we have constructed persistent homology, rather carefully. It only took us 35 hours!

The construction makes equal sense for filtrations of simplicial complexes and filtrations of topological spaces. In the latter case, we use singular homology.

Interpretation of Persistent Homology

Key geometric idea: Not only can we "count holes" in each simplical complex in a filtration, individually, we can track holes in consistent way across the whole filtration.

Example: Consider this filtration:



$\mathrm{Barc}(H_1(F)) = \{[1,6), [4,5)\}$.

The interval $[1,6)$ corresponds to the first hole to appear.
The interval $[4,5)$ corresponds to the second hole.

In general, for any filtration F:

- Each interval $[a,b) \in \mathrm{Barc}(H_i(F))$ corresponds to a hole in the filtration (subtlety: this correspondence is not unique)
- a is the index at which this hole forms ("is born")
- b is the index at which this hole closes up ("dies")