AMAT 584 Lecture 3 1/27/20

Last time: Simplices
Today: Simplicial complexes

Review: For $k \geq 0$, a *k-simplex* in $\mathbb{R}^n$ is the convex hull of $k+1$ points in general position.
If these points are $x_0, \ldots, x_k$, we write the simplex as $[x_0, \ldots, x_k]$.

- *General position* means these do not lie on a $(k-2)$-dim. affine subspace.

- For $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}^n$ finite, the *convex hull* of X is
$$\mathrm{Conv}(X) = \{c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \mid c_i \geq 0, \sum_{i=1}^n c_i = 1\}$$

Definition: $X \subset \mathbb{R}^n$ is convex if $\forall x, y \in X$, X contains the line segment connecting x and y.

Remark: $\mathrm{Conv}(X)$ is the smallest convex set containg X.

A 0-simplex is a point. (Technically, a singleton set).
A 1-simplex is a line segment
A 2-simplex is a triangle
A 3-simplex is a tetrahedron
⋮

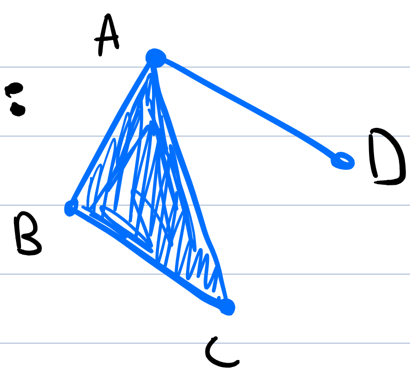
Definition: A **face** of a simplex $[x_0, \dots x_k]$ is the simplex spanned by a non-empty subset of $\{x_1, \dots, x_k\}$.

Exercise: List all the faces of the simplex. $[0,1] \subset \mathbb{R}$.

Definition: A (geometric) simplical complex is a set S of simplices in $\mathbb{R}^n$ (for some fixed n) such that

1. each face of a simplex in S is contained in S
2. the intersection of two simplices in X is a face of each of them (if non empty).

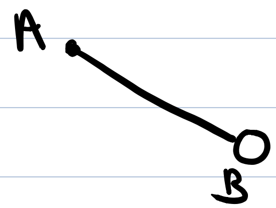
Example:



Let $A, B, C, D \in \mathbb{R}^2$ be as shown

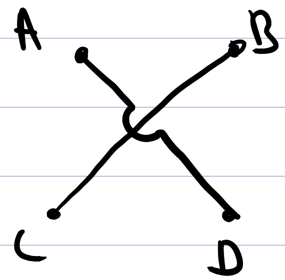
This illustrates the simplicial complex $\{[A], [B], [C], [D], [A,B], [A,C], [B,C], [A,D], [A,B,C]\}$.

Example: For $A, B \in \mathbb{R}^2$ as shown,



$\{[A], [A, B]\}$ is <u>not</u> a simplical complex: property 1 is violated.

Example: For $A, B, C, D \in \mathbb{R}^2$ as shown,



$\{[A], [B], [C], [D], [A, D], [B, C]\}$ is <u>not</u> a simplial complex: Property 2. is violated.

Remark: As the examples show, a simplical complex is specified by set of points in $\mathbb{R}^n$, together with a collection of finite subsets of these satisfying certain conditions.

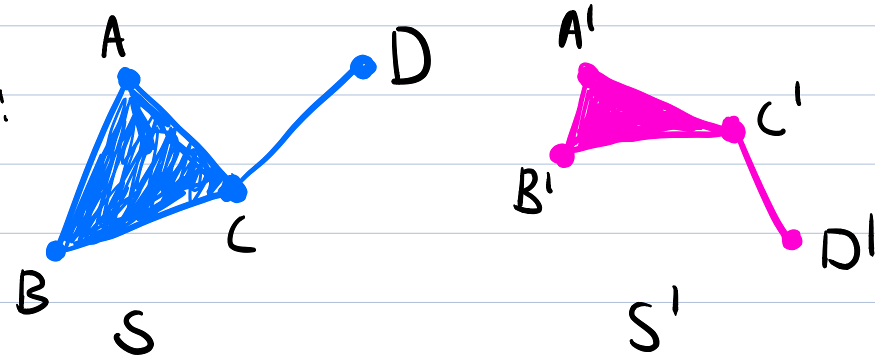
We say X is <u>k-dimensional</u> if the largest dimension of a simplex in X is k.

Definition: For S a simplicial complex, we call the union of the simplices in S the <u>geometric realization</u> of S, and denote this $|S|$.

Abstract Simplicial Complexes

Motivation: It turns out that up to homeomorphism, $|S|$ doesn't depend on the position of the 0-simplices of S.

Example:



$|S| \cong |S'|$. (Recall that $\cong$ means "is homeomorphic to").

Let's make this precise:

Proposition: Let S and S' be simplicial complexes, and suppose there is a bijection f from the 0-simplices of S to the 0-simplices of S' such that

$$[x_0, \dots, x_k] \in S \text{ iff } [f(x_0), \dots, f(x_k)] \in S'.$$

Then $|S| \cong |S'|$.

Def: An (abstract) simplicial complex is a set X of non-empty finite sets such that if $\sigma \in X$ and $\tau \subset \sigma$ is non-empty, then $\tau \in X$.

Given a geometric simplicial complex Y, we obtain an abstract simplicial complex $\mathrm{Abs}(Y)$ by

$$\mathrm{Abs}(Y) = \{\{x_0, \dots, x_n\} \mid [x_0, \dots, x_1] \text{ a simplex in } Y\}.$$

In this sense, abstract simplicial complexes generalize geometric ones.

This connection motivates the following notation:
A set $\{a_0, \dots, a_k\}$ in an abstract simplicial complex of size $k+1$ is called a (k-)<u>simplex</u>, and is denoted $[a_0, \dots, a_k]$.

An abstract simplicial complex is called k-dimensional if the largest dimension of a simplex is k

<u>Example</u>: $X = \{[a], [b], [c], [a,b], [b,c], [a,c]\}$ is a 1-dimensional simplicial complex.

<u>Non-example</u>: $X = \{[a,b], [b,c], [a,c]\}$ is not an abstract simplicial complex.

<u>The connection between abstract and geometric simplicial complexes.</u>

We just described a map Abs from geometric simplicial complexes to abstract simplical complexes.

We now describe a map in the other direction.

This allows us to think of an abstract simplicial complex in geometric terms.

$$\begin{tikzcd} \text{Geo. Simp. Complexes} \arrow[r, bend left, "\text{Abs}"] & \text{Abs. Simp. Complexes} \arrow[l, bend left, dashed, "\text{Geo}"] \end{tikzcd}$$

Let X be a finite abstract simplicial complex. Order the 0 simplices of X arbitrarily and denote them $[x_1], \ldots, [x_m]$.

Letting $e_i \in \mathbb{R}^m$ be given by $(0, \ldots, 0, \underset{\substack{\uparrow \\ i^{\text{th}} \text{ entry.}}}{1}, 0, \ldots, 0)$

we define

$$\mathrm{Geo}(X) = \{[e_{j_0}, e_{j_1}, \ldots, e_{j_k}] \mid \underset{X}{\underset{\cap}{[x_{j_0}}}, x_{j_1}, \ldots, x_{j_k}] \}.$$