AMAT 584 Lec. 4 1/29/19

Today: Abstract simplicial complexes.

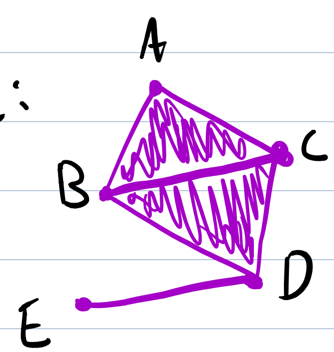
Definition: A (geometric) simplical complex is a set X of simplices in $\mathbb{R}^n$ (for some fixed n) such that

1. each face of a simplex in X is contained in
2. the intersection of two simplices in X is a face of each of them (if non empty).

The dimension of a simplicial complex is the largest dimension of one of its simplices.

Note: We say a k-simplex has dimension k

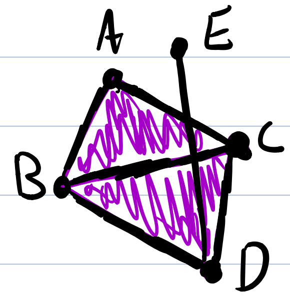
Example:



For $\{A, B, C, D, E\} \subset \mathbb{R}^2$ as shown,

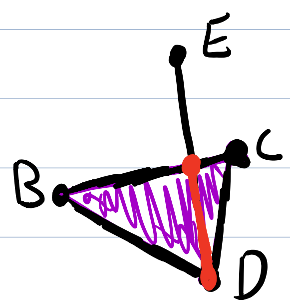
$X = \{[A], [B], [C], [D], [E], [A,B], [B,C], [A,C],$
$[B,D], [C,D]$
$[A,B,C], [B,C,D]$ is a simplicial complex

Example: For $\{A, B, C, D, E\} \subset \mathbb{R}^2$ as shown,



The same list of simplices as shown above is not a simplicial complex.

Property 2 is violated. For example $[B,C,D] \cap [D,E]$ is not a face of $[B,C,D]$ or $[D,E]$

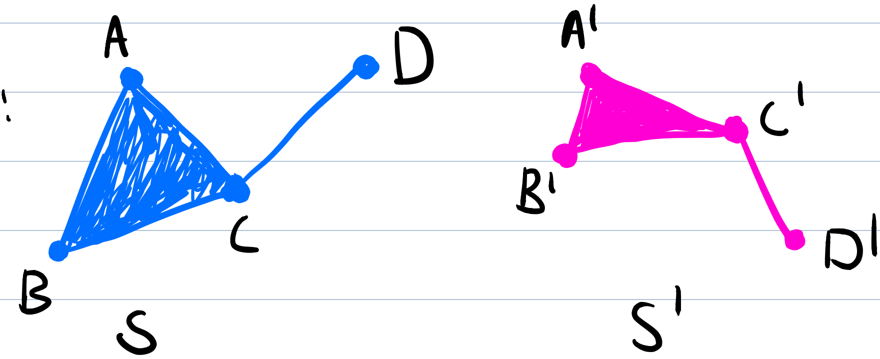


Def: The geometric realization $|X|$ of a simplicial complex X is the union of its simplices.

Abstract Simplicial Complexes

Motivation: It turns out that up to homeomorphism, $|S|$ doesn't depend on the position of the 0-simplices of S.

Example:



$|S| \cong |S'|$. (Recall that $\cong$ means "is homeomorphic to").

Let's make this precise:

Definition: Geometric simplicial complexes X and X' are *isomorphic* if there is bijection f from the 0-simplices of X to the 0-simplices of X' such that

$$[x_0, \dots, x_k] \in X \text{ iff } [f(x_0), \dots, f(x_k)] \in X'$$

Then $|X| \cong |X'|$.

This suggests that the topological structure of a simplicial complex can be specified in more abstract terms.

Def: An **(abstract) simplicial complex** is a set X of non-empty finite sets such that if $\sigma \in X$ and $\tau \subset \sigma$ is non-empty, then $\tau \in X$.

Given a geometric simplicial complex Y, we obtain an abstract simplicial complex $\mathrm{Abs}(Y)$ by

$$\mathrm{Abs}(Y) = \{\{x_0, \dots, x_n\} \mid [x_0, \dots, x_1] \text{ a simplex in } Y\}.$$

In this sense, abstract simplicial complexes generalize geometric ones. This motivates the following notation:

A set $\{a_0, \dots, a_k\}$ in an abstract simplicial complex of size $k+1$ is called a **(k-)simplex** (or a **simplex of dimension k**) and is denoted $[a_0, \dots, a_k]$.

An abstract simplicial complex is called **k-dimensional** if the largest dimension of a simplex is k.

Example: $X = \{[a], [b], [c], [a,b], [b,c], [a,c]\}$ is a 1-dimensional simplicial complex.

Non-example: $X = \{[a,b], [b,c], [a,c]\}$ is not an abstract simplicial complex.

The connection between abstract and geometric simplicial complexes.

We just described a map Abs from geometric simplicial complexes to abstract simplical complexes.

We now describe a map in the other direction.

This allows us to think of an abstract simplicial complex in geometric terms.

$$\begin{tikzcd}
\text{Geo. Simp. Complexes} \arrow[r, bend left, "\text{Abs}"] & \text{Abs. Simp. Complexes} \arrow[l, bend left, dashed, "\text{Geo}"]
\end{tikzcd}$$

<u>Def</u>: For X an abstract simplicial complex, let $\bigcup_{\sigma \in X} \sigma$ be called the <u>vertex</u> <u>set</u> of X.

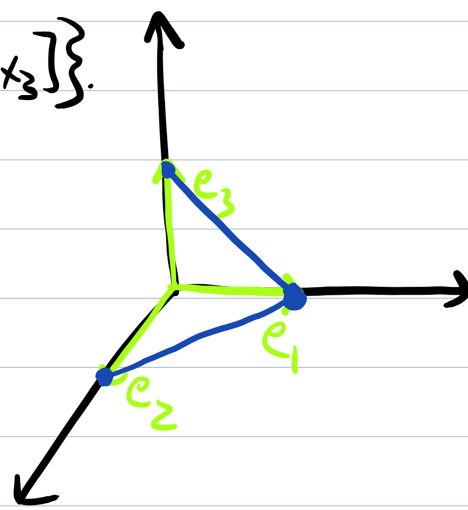
Let X be a <u>finite</u> abstract simplicial complex. Write the vertex set of X as $\{x_1, \ldots, x_m\}$, choosing the order arbitrarily.

Letting $e_i \in \mathbb{R}^m$ be given by $(0, \ldots, 0, 1, 0, \ldots, 0)$
we define $\hspace{5em}$ i^{th} entry.

$$\mathrm{Geo}(X) = \{[e_{j_0}, e_{j_1}, \ldots, e_{j_k}] \mid [x_{j_0}, x_{j_1}, \ldots, x_{j_k}] \in X\}$$

<u>Example</u>: Let $X = \{[a], [b], [c], [a,b], [a,c]\}$

Write $a = x_1$
$b = x_2$
$c = x_3$, then
$X = \{[x_1], [x_2], [x_3], [x_1, x_2], [x_1, x_3]\}$.

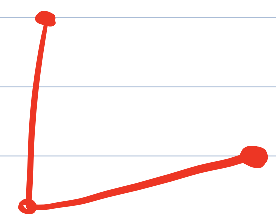


$\mathrm{Geo}(X) =$
$\{[e_1], [e_2], [e_3], [e_1, e_2],$
$[e_1, e_3]\}$.

This is a collection of simplices in $\mathbb{R}^3$.

<u>Note</u>: Often we can find a geometric simplicial complex isomorphic to $\mathrm{Geo}(X)$ and living in a lower dimensional space.

For instance, for X as in the last example, the geometric simplicial complex in $\mathbb{R}^2$ shown below is isomorphic to $\mathrm{Geo}(X)$:



Def: A <u>morphism</u> $f: X \to Y$ between abstract simplicial complexes is a function $f: V(X) \longrightarrow V(Y)$ such that for each simplex $\sigma \in X$, $f(\sigma) \in Y$.

Note: Composition of morphisms is well defined.

Def: An <u>isomorphism</u> of abstract simplicial complexes is an invertible morphism.