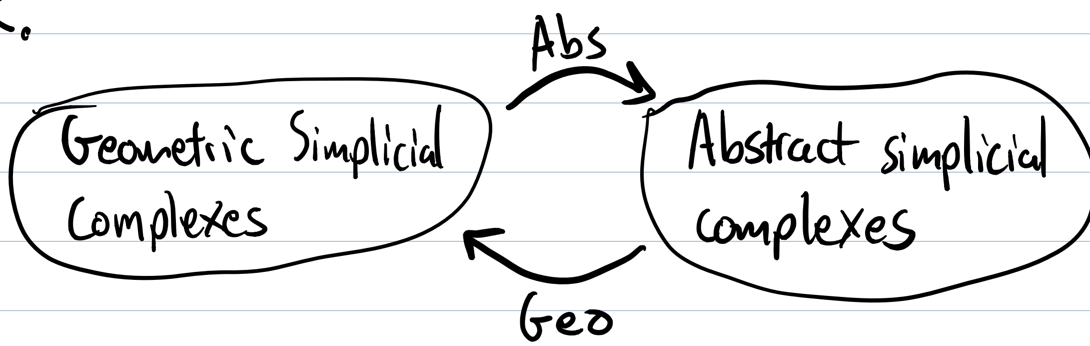
AMAT 584 Lec. 5 1/31/2020

Today: The relationship between geometric and abstract simplicial complexes.

Review:

Def: An (abstract) simplicial complex is a set X of non-empty finite sets such that if $\sigma \in X$ and $\tau \subset \sigma$ is non-empty, then $\tau \in X$.

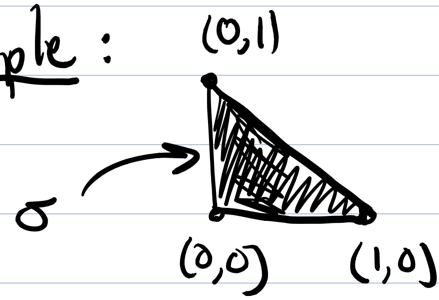
If $\sigma \in X$ and $|\sigma| = k+1$ then σ is called a (k-) simplex, or a simplex of dimension k.



For X a geometric simplicial complex, we defined an abstract simplicial complex $Abs(X)$.

Given a geometric simplex $\sigma \subset \mathbb{R}^n$, let corners($\sigma$) denote the set of its corners.

Example:



$\text{corners}(\sigma) = \{(0,0), (1,0), (0,1)\}$.

Then $\text{Abs}(X) = \{\text{corners}(\sigma) \mid \sigma \in X\}$.

From an Abstract to a Geometric Simplicial Complex

Def: For X an abstract simplicial complex, let $\bigcup_{\sigma \in X} \sigma$ be called the *vertex set* of X.

Let X be a finite abstract simplicial complex. Write the vertex set of X as $\{x_1, \ldots, x_m\}$, choosing the order arbitrarily.

Letting $e_i \in \mathbb{R}^m$ be given by $(0, \ldots, 0, 1, 0, \ldots, 0)$ we define

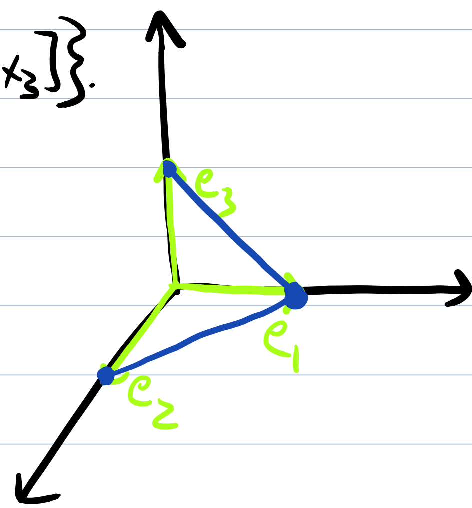
i^{th} entry.

$$\text{Geo}(X) = \{[e_{j_0}, e_{j_1}, \ldots, e_{j_k}] \mid [x_{j_0}, x_{j_1}, \ldots, x_{j_k}] \in X\}$$

Example: Let $X = \{[a], [b], [c], [a,b], [a,c]\}$

Write $a = x_1$
$b = x_2$
$c = x_3$, then

$X = \{[x_1], [x_2], [x_3], [x_1, x_2], [x_1, x_3]\}$.

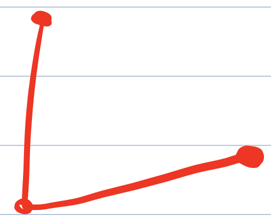


$Geo(X) = \{[e_1], [e_2], [e_3], [e_1, e_2], [e_1, e_3]\}$.

This is a collection of simplices in $\mathbb{R}^3$.

Note: Often we can find a geometric simplicial complex isomorphic to $Geo(X)$ and living in a lower dimensional space.

For instance, for X as in the last example, the geometric simplicial complex in $\mathbb{R}^2$ shown below is isomorphic to $Geo(X)$:

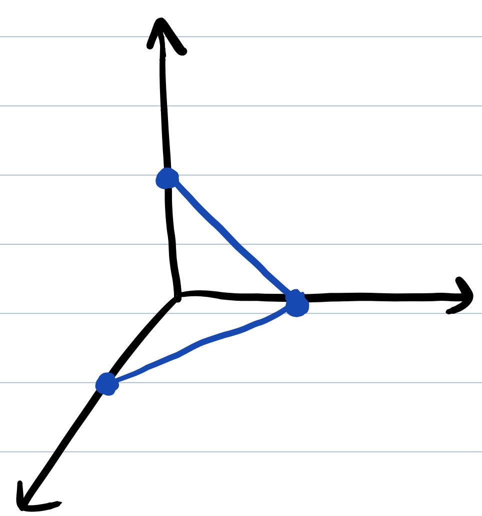


<u>Definition</u>: If X is a finite abstract simplicial complex, then we can define the <u>geometric realization</u> of X, denoted $|X|$, by $|X| = \underbrace{|\mathrm{Geo}(X)|}_{\text{union of all the simplices in } X.}$.

Note that $|X|$ is a topological space.

$$\text{Abstract Simp. Complex} \xrightarrow{\text{Geo}} \text{Geo. Simp. Complex} \xrightarrow{|\cdot|} \text{Top. Space}$$

For instance, in the previous example, $|X|$ is the blue "V" shown in the figure.



Interpretation of geometric realization of an abstract simplicial complex, in terms of gluing simplices:

For most abstract simp. complexes X, $|X|$ is hard to visualize directly, so the following characterization can be helpful for intuition:

Given an abstract simplicial complex X, $|X|$ is constructed as follows:

- We start with a discrete set, with one point for each 0-simplex in X.
- For each 1-simplex, we glue in a line segment
- For each 2-simplex, we glue in a triangle

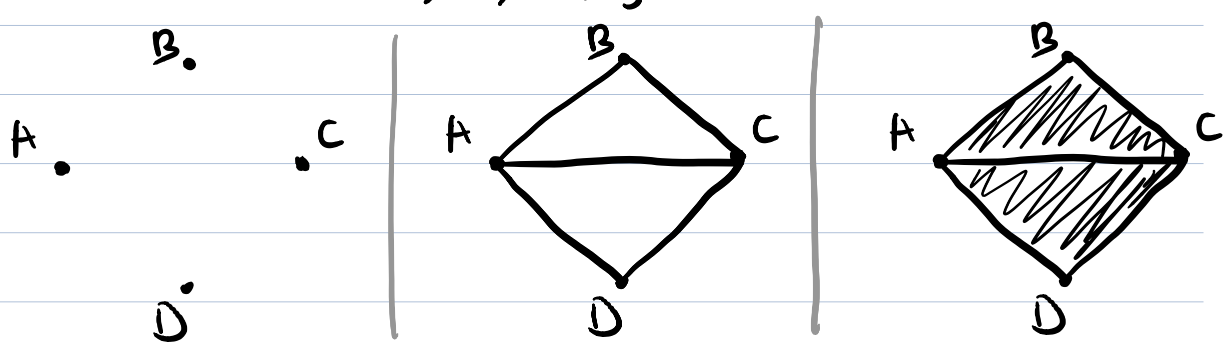
⋮

etc.

This characterization of $|X|$ can be made formal, of course, but we won't bother with that here.

Example:

Let $X = \{[A],[B],[C],[D],[A,B],[A,C],[B,C],[B,D],[C,D],$
$[A,B,C],[B,C,D]\}$.



Definition: An abstract simplicial complex of dimension at most 1 is called an (undirected) graph.

Note: This definition is equivalent to the standard definition of a graph that we saw in TDA.

Remark: Our definition of geometric realization of abstract simplicial complexes extends without difficulty to the case where X has an infinite number of simplices.

(One way to carry out this extension would be to define geometric simplicial complexes for possibly infinite-dimensional vector spaces. There is no difficulty in this.)

Fact: For any finite geometric simplicial complex X, $\mathrm{Geo}(\mathrm{Abs}(X))$ is isomorphic to X.

Moreover, we can define a notion of isomorphism between abstract simplicial complexes, and we also have that for any finite abstract simplicial complex X, $\mathrm{Abs}(\mathrm{Geo}(X))$ is isomorphic to X.

In this sense, abstract and geometric simplicial complexes are equivalent.

Simplicial maps In what follows, all simplicial complexes will be abstract.

Recall our notation that $V(X)$ denotes the vertex set of a simplicial complex X.

Let X and Y be simplical complexes.

A simplicial map $f: X \to Y$ is a function $f: V(X) \to V(Y)$