AMAT 584 Lec 6

Today: - Simplicial Maps

Next time: - The connection between Geometric + Abstract simplicial complexes
- Euler Characteristic.

Review:

Def: An (abstract) simplicial complex is a set X of non-empty finite sets such that if $\sigma \in X$ and $\tau \subset \sigma$ is non-empty, then $\tau \in X$.

The vertex set of X is $V(X) = \bigcup_{\sigma \in X} \sigma$.

Simplicial maps In what follows, all simplicial complexes will be abstract.

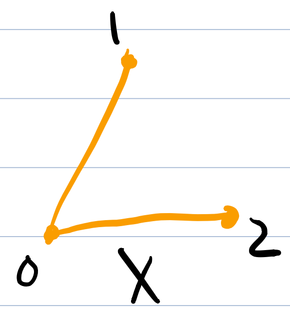
Let X and Y be simplical complexes.

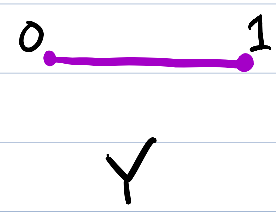
A simplicial map $f: X \to Y$ is a function $f: V(X) \to V(Y)$ such that if $\sigma \in X$ then $f(\sigma) \in Y$.

Notation: $f(\sigma) = \{f(x) \mid x \in \sigma\} \subset V(Y)$.

Example: $X = \{[0],[1],[2],[0,1],[0,2]\}$

$Y = \{[0],[1],[0,1]\}$.





$V(X) = \{0,1,2\}$
$V(Y) = \{0,1\}$

Define $f: X \to Y$ by

$f(0) = f(2) = 0$
$f(1) = 1$.

Let's check that this is simplicial:

$f([0]) = [0] \in Y$
$f([1]) = [1] \in Y$
$f([2]) = [0] \in Y$
$f([0,1]) = [0,1] \in Y$
$f([0,2]) = [0] \in Y$.

So f is indeed simplicial.

__Exercise__: Consider $g: V(X) \to V(X)$,

$g(0) = 1$
$g(1) = 2$
$g(2) = 0$.

Is g a simplicial map from X to X?

__Ans__: No, because $g([0,1]) = \underset{\{1,2\}.}{\underset{\shortparallel}{[1,2]}} \notin X$.

The most important simplicial maps are the __inclusions__:

__Def__: A __subcomplex__ of a simplicial complex X is a simplicial complex Y such that if $\sigma \in Y$, then $\sigma \in X$.

In other words, a subcomplex is a __subset__ that is itself a simplicial complex.

If Y is a subcomplex of X, we write $Y \subset X$.

__Example__: For X, Y as in the first example of this lecture, we have $Y \subset X$.

Easy fact: If $Y \subset X$, then $V(Y) \subset V(X)$, and the inclusion $j: V(Y) \hookrightarrow V(X)$ is a simplicial map from Y to X.

Aside on inclusions

If $S \subset T$ are any sets, there is an inclusion function $j: S \to T$, given by $j(x) = x$.

A special case of the inclusion is the identity map Id_X on a simplicial complex X.

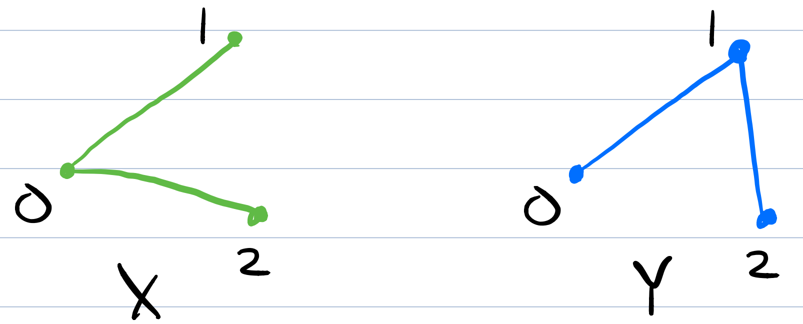
Composition of Simplicial maps

If $f: X \to Y$ and
$g: Y \to Z$ are simplicial maps,
then we have a composite map $g \circ f: X \to Z$.

Exercise: Check this.

Def:
A simplicial map $f: X \to Y$ is an **isomorphism** if it has an inverse simplicial map $g: Y \to X$, i.e. $g \circ f = Id_X$, $f \circ g = Id_Y$.

<u>Example</u> $X = \{[0],[1],[2],[0,1],[0,2]\}$
$Y = \{[0],[1],[2],[0,1],[1,2]\}$



$f: X \to Y$, $f(0) = 1$, $f(1) = 2$, $f(2) = 0$ is an isomorphism, with inverse

$f^{-1}: Y \to X$, $f^{-1}(0) = 2$, $f^{-1}(1) = 0$, $f^{-1}(2) = 1$.

<u>Geometric realization of simplicial maps</u>

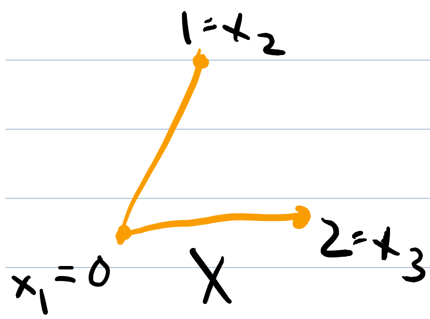
A simplicial map $f: X \to Y$ induces a <u>continuous</u> map on the geometric realizations

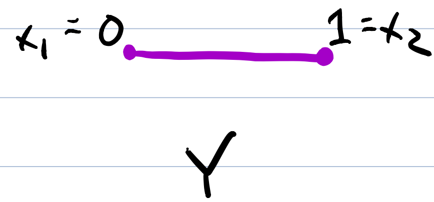
$|f|: |X| \to |Y|$.

Consider the example of earlier:

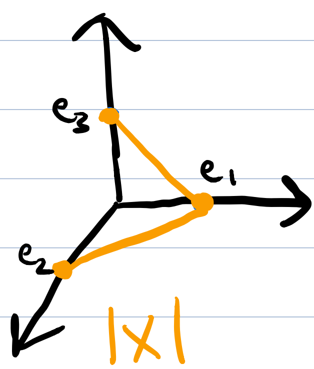
$X = \{[0],[1],[2],[0,1],[0,2]\}$, $f(0) = f(2) = 0$
$f(1) = 1$

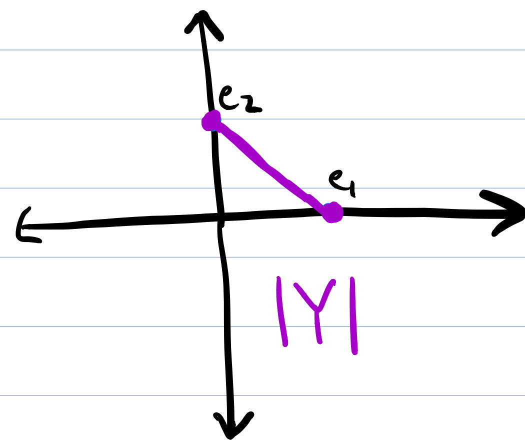
$Y = \{[0],[1],[0,1]\}$.





Recall $|X|$ is defined by $|X| = |Geo(X)|$,
i.e. $|X|$ is the union of the simplices in $Geo(X)$.





<u>To define $|f|$:</u>

1) f induces a map on $|f|$ on the 0-simplices of $Geo(f)$.

In the example above, $|f|(e_1) = |f|(e_3) = e_1$.
$|f|(e_2) = e_2$

2) Exend the definition of $|f|$ to each simplex in $Geo(X)$, as follows

If $\sigma = [x_0, \ldots, x_k] = \{c_0x_0 + c_1x_1 + \cdots + c_kx_k \mid c_i \geq 0, \sum_i c_i = 1.\}$

Then $|f|(c_0x_0+c_1x_1+\cdots+c_kx_k)=c_0|f|(x_0)+\cdots+c_k|f|(x_k)$
$\in |Y|.$

3) Check that the maps on simplices agree on their intersection, so that they induce a map $|f|:|X|\to|Y|$