AMAT 584 Lec 7

Today: Geometric realization of simplicial maps
Euler Characteristic

Recall: For (abstract) simplicial complexes X and Y, a simplicial map $f: X \to Y$ is a function $f: V(X) \to V(Y)$ such that $\forall \sigma \in X$, $f(\sigma) \in Y$.

As mentioned at the end of class last time, a simplicial map $f: X \to Y$ induces a continuous map $|f|: |X| \to |Y|$.

Geometric Realizations of X and Y

We now explain how $|f|$ is defined.

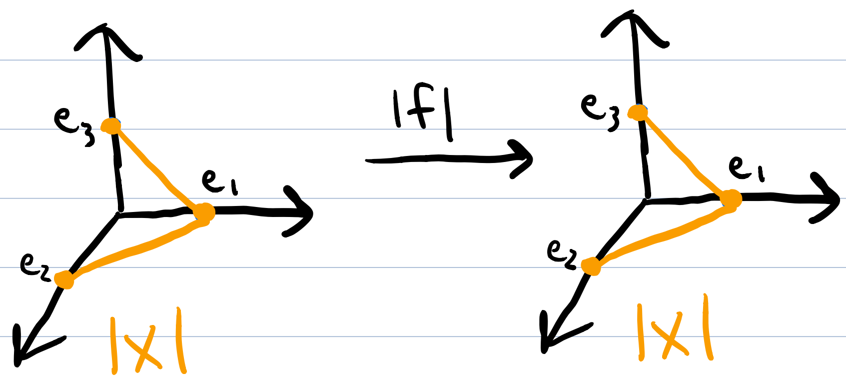
Consider the example:

$$X = \{[1],[2],[3],[1,2],[1,3]\},$$

$f: X \to X$, $\quad f(1) = f(3) = 1$

$\quad\quad\quad\quad f(2) = 2.$

Recall $|X|$ is defined by $|X| = |Geo(X)|$, i.e., $|X|$ is the union of the simplices in $Geo(X)$.



We want to define $|f|$ so that $|f|(x)=x$ for $x\in[e_1,e_2]$, and and $|f|(x)=e_1$ for $x\in[e_1,e_3]$.

To define $|f|$ in general, we proceed in 3 steps:

Here, we assume that X and Y are finite, though this assumption can be avoided.

1) f induces a map on $|f|$ from the 0-simplices of $Geo(X)$ to the 0-simplices of $Geo(Y)$.

In the example above, $|f|(e_1)=|f|(e_3)=e_1$.
$|f|(e_2)=e_2$

2) Exend the definition of $|f|$ to each simplex $\sigma\in Geo(X)$ by linearity, i.e., as follows:

$$\text{If } \sigma=[e_1,\dots,e_k]=\left\{c_1e_1+c_1e_1+\cdots+c_ke_k \,\middle|\, c_i\geq 0,\ \sum c_i=1\right\}$$

$$\text{Then } |f|(ce_1+ce_2+\cdots+c_ke_k)=c_1|f|(e_1)+\cdots+c_k|f|(e_k).$$

If $\sigma = [e_1, e_3]$ in the example above, then

$$|f|\left(\tfrac{1}{2}e_1 + \tfrac{1}{2}e_2\right) = \tfrac{1}{2}|f|(e_1) + \tfrac{1}{2}|f|(e_2)$$
$$= \tfrac{1}{2}e_1 + \tfrac{1}{2}e_1 = e_1.$$

3) Check that the maps on simplices agree on their intersection, so that they induce a map $|f|: |X| \to |Y|$.

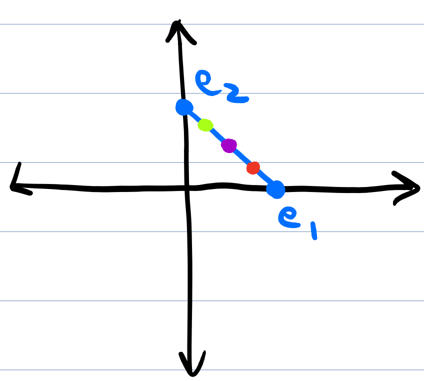
(and by standard ideas in point-set topology, this map is continuous).

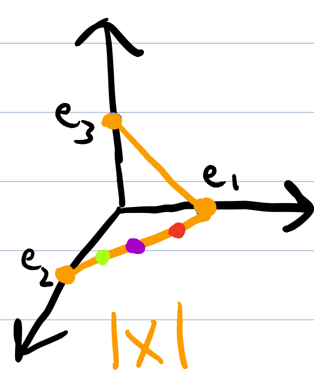
For $\sigma = [e_1, e_2]$ and $\tau = [e_1, e_3]$ in the example above,
$|f|: \sigma \to |Y|$ maps e_1 to e_1, and
$|f|: \tau \to |Y|$ maps e_1 to e_1, so these maps agree on their intersection.

Geometric realization of inclusion maps

As we said last lecture, the most important type of simplicial map is an inclusion.

$j: Y \hookrightarrow X$. For such j, $|j|: |Y| \to |X|$ is an embedding.





Via this embedding, we can think of $|Y|$ as a subspace of $|X|$.

(Recall: A subspace of a metric space is a subset, regarded as a metric space via restriction.
Here the metric is the Euclidean metric
Or more generally, a subspace of a topological space is a subset endowed with the **subspace topology**).

"Functoriality Properties" of Induced Maps on Geometric Realizations

Proposition:

1) For any (abstract) simplicial complex X, $|Id_X| = Id_{|X|}$.
that is, geometric realization sends identity maps to identity maps

2) For any simplicial maps $f: X \to Y$, $g: Y \to Z$, $|g \circ f| = |g| \circ |f|$
that is, geometric realization respects composition

Corollary: If $f: X \to Y$ is an isomorphism of abstract simplicial complexes, then $|f|: |X| \to |Y|$ is a homeomorphism.

Proof: $f^{-1} \circ f = Id_X \Rightarrow |f^{-1} \circ f| = |Id_X| \overset{\text{Prop}}{\Rightarrow} |f^{-1}| \circ |f| = Id_{|X|}$
Similarly, $f \circ f^{-1} = Id_Y \Rightarrow |f| \circ |f^{-1}| = Id_{|Y|}$.

Thus, $|f|$ is a continuous bijection, and its inverse is $|f^{-1}|$ which is also continuous. $\blacksquare$

<u>Remarks on the big picture:</u>
Via geometric realization, abstract simplicial complexes and simplicial maps be thought of as discrete models of topological spaces.

Since they are discrete, they are quite convenient for computation.

<u>Definition</u>: A topological space is <u>triangulable</u> if it is homeomorphic to the geometric realization of a simplicial complex.

Not all "nice" topological spaces are triangulable

For example, it is known that not all 4-D manifolds are triangulable.

And not all topological spaces are even <u>homotopy equivalent</u> to a simplicial complex.

But as a rule, "nice spaces" are homotopy equivalent to simplicial complexes. One can make this precise using the language of CW-complexes, for example.

Euler characteristic

Let's consider the following simplicial complexes

For each example let's consider the following quantity:

#0 simplices − #1 simplices + #2-simplices − #3-simplices.

This is called the Euler characteristic.