AMAT 584 Lec 8 2/7/20

Today: Euler characteristic
Simplicial complexes constructed from data

Euler characteristic

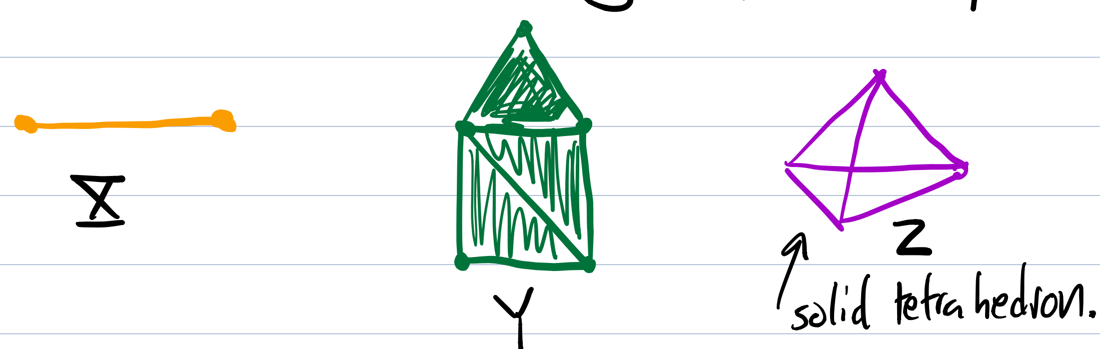
For X a finite simplicial complex and $k \in \mathbb{N}$, let $n_k(X)$ denote the number of k-simplices of X.

Definition: The Euler characterisic of X is

$$\sum_{i=0}^{\infty} (-1)^i n_k(X) = n_0(X) - n_1(X) + n_2(X) - n_3(X) + \cdots$$

Since X is finite, this is actually a finite sum

Examples:
Let's consider the following simplicial complexes

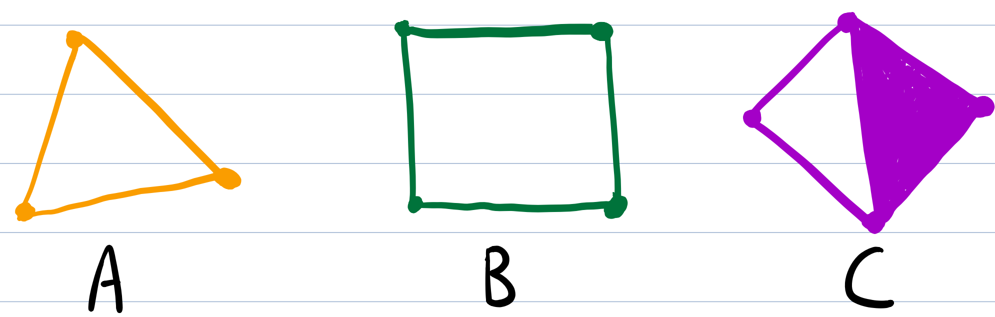


Note that the geometric realization of each of these simplicial complexes is contractible, i.e., homotopy equivalent to a point.

$\chi(X) = 2 - 1 = 1$
$\chi(Y) = 5 - 7 + 3 = 1$
$\chi(Z) = 4 - 6 + 4 - 1 = 1$

X, Y, and Z have the same Euler Characteristic!

Now let's consider 3 more examples:



Note that the geometric realization of each of these simplicial complexes is homotopy equivalent to a circle.

$\chi(A) = 3 - 3 = 0$
$\chi(B) = 4 - 4 = 0$
$\chi(C) = 4 - 5 + 1 = 0$

<u>Theorem</u>: If X and Y are simplicial complexes whose geometric realizations are homotopy equivalent, then $\chi(X) = \chi(Y)$.

The theorem tells us that, in a precise sense, the Euler characteristic is a <u>topological</u> <u>invariant</u>.

<u>Contrapositive</u>: Equivalently, if $\chi(\mathbb{X}) \neq \chi(Y)$ then then $|\mathbb{X}|$ and $|Y|$ are not h.e.!

<u>Corollary</u>: A circle is not contractible (i.e., no h.e. to a point).

<u>Fact from last semester</u>:
Homotopy equivalence is transitive, i.e., $M \simeq N$ and $N \simeq P$ implies $M \simeq P$.

$\simeq$ means "is homotopy equivalent to"

<u>Proof</u>: For $\mathbb{X}$ and A above, $|\mathbb{X}| \simeq *$, and $|A| \simeq S^1$ ← circle

notation for a point.

$\chi(\mathbb{X}) = 1 \neq 0 = \chi(A)$, so $|\mathbb{X}| \not\simeq |A|$.

If it were true that $* \simeq S^1$, then

$|\mathbb{X}| \simeq * \simeq S^1 \simeq |A|$, which implies $|\mathbb{X}| \simeq |A|$ by transitivity, contradicting the above.

Thus, it cannot be true that $* \simeq S^1$. $\blacksquare$

<u>Remark</u>: One doesn't really need the Euler characteristic to show that $* \simeq S^1$. Ideas from TDA I suffice.

However, the same argument can be used to show that many other pairs of spaces are not homotopy equivalent.

e.g. Sphere and torus are not h.e.

<u>Context</u>: The idea of the Euler characteristic dates back to the mid-1700's.

Euler and Decartes independently discovered that if X is the surface of a convex polyhedron, then $\chi(X) = 2$.

This is one of the early major developments of topology.

The Euler characteristic can be defined in much more generality, and plays a central role in geometry and topology.

It is also useful in TDA, as we will discuss.

Idea behind the proof of the topological invariance of the E.C.

This will be informal, we will be more precise later in the course.

Let $B_k(X)$ denote the # of k-dimensional holes in X.

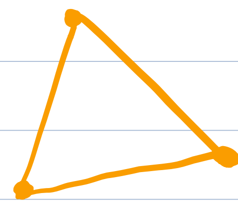
<u>Recall:</u> A 0-D hole is a connected component
A 1-D hole is one you can see through
A 2-D hole is a hollow space.
⋮

Fact: (to be explained more carefully later).
If $|X| \simeq |Y|$ then $B_k(X) = B_k(Y)$ $\forall k \geq 0$.

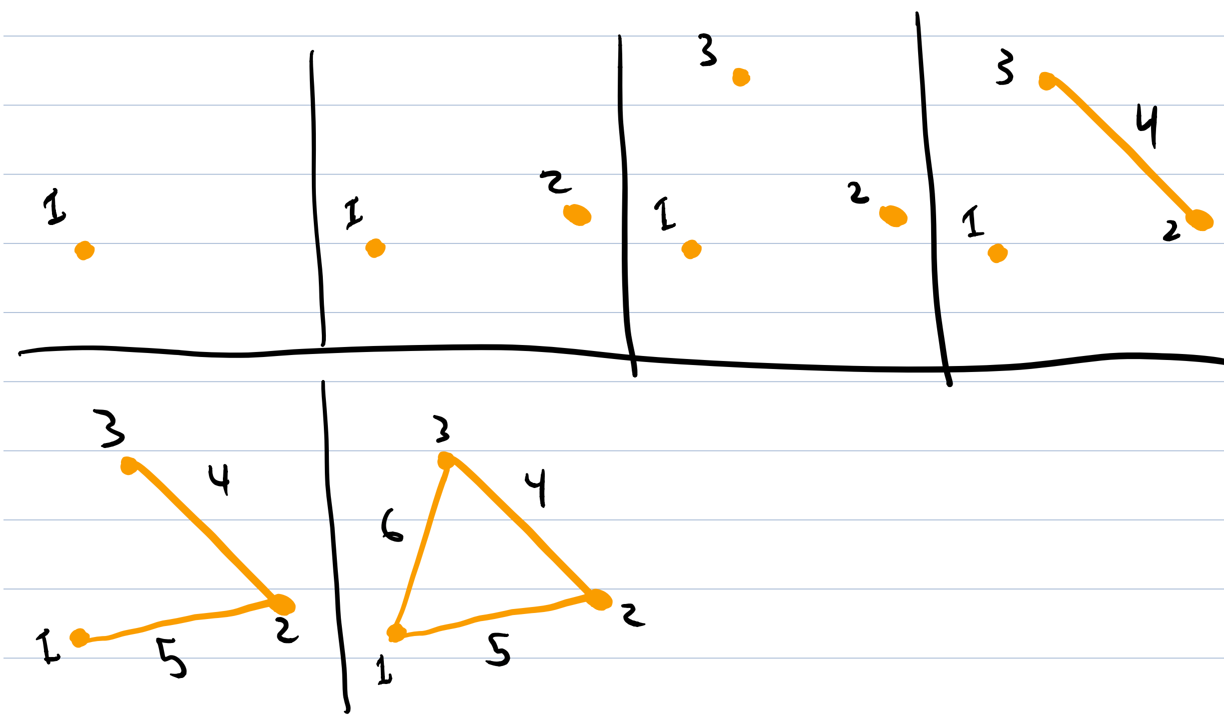
Claim: $\chi(X) = B_0(X) - B_1(X) + B_2(X) - \cdots$
$$= \sum_{i=0}^{\infty} (-1)^i B_i(X).$$

If this is true, then by the fact above, $\chi(X) = \chi(Y)$ whenever $|X| \simeq |Y|$.

To understand why the claim is true, imagine constructing X by adding 1-simplex at a time, in order of increasing dimension.



A



Every time I a k-simplex, I either create a new k-dim. hole, or close of an existing k-1 dimensional hole.

Each k-simplex σ closing a hole can be paired with a $(k-1)$-dimensional simplex τ which created the hole σ closes.

For example, above, we can pair 4 with 3. The pair (τ, σ) doesn't contibute to the E.C because the simplices are in neighboring dimension.

So $$\chi(X) = \sum_{i=0}^{\infty} (\#\text{ unpaired simples})(-1)^i$$

$$= \sum_{i=0}^{\infty} (-1)^i B_i(X).$$

Because the unpaired k-simplices correspond to the k-dimensional holes in X.