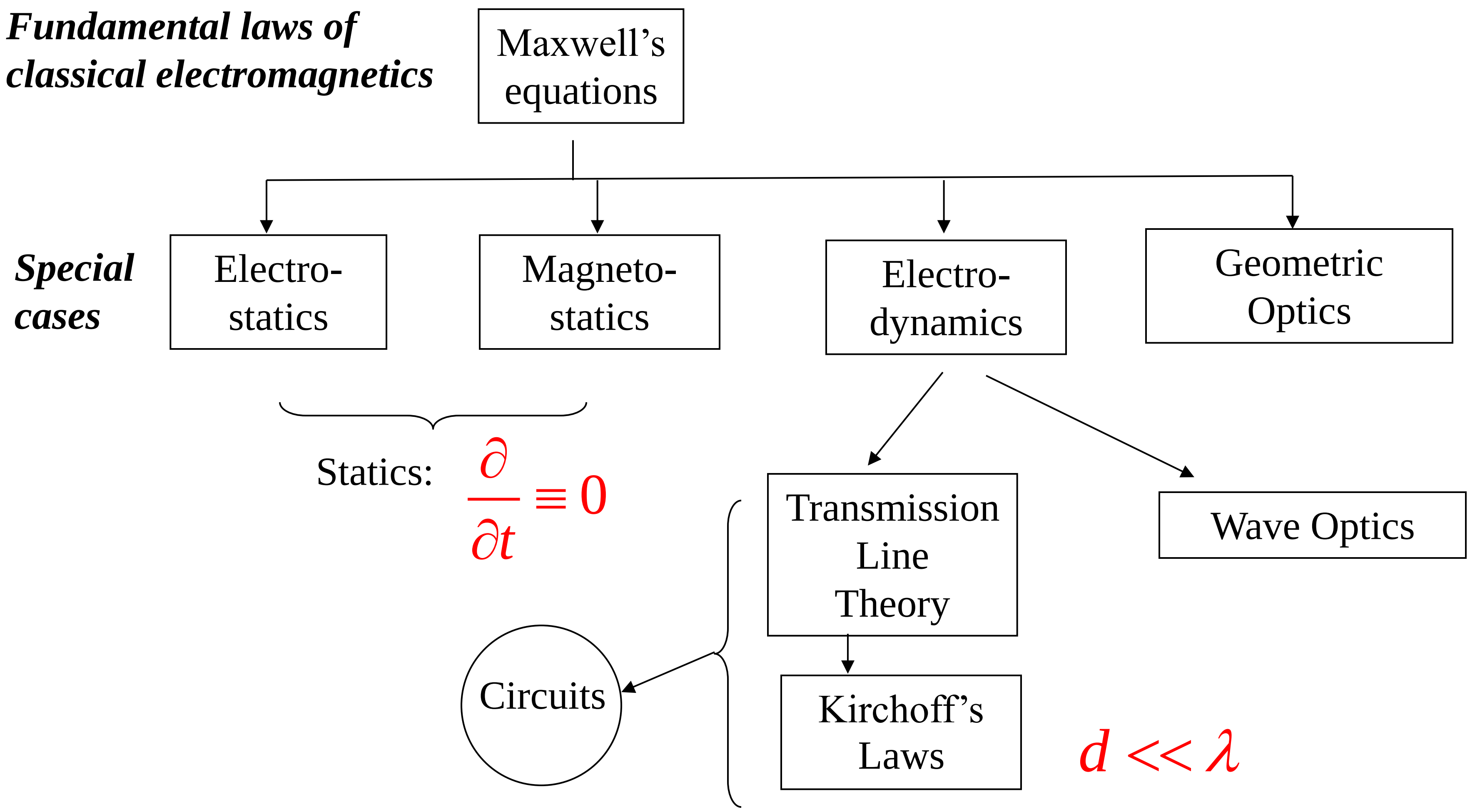
Lecture contents

Review: Few concepts from physics

- **Electric field**
  - **Coulomb law, Gauss law, Poisson equation, dipole, capacitor**
  - **Conductors and isolators 1**
  - **Electric current**
  - **Dielectric constant**

Overview of Electromagnetics



Few concepts from Electrostatics: Coulomb law

- Charge
  - Charges interact →
  - Quantized e = 1.602x10^{-19} C
  - Conserved

Coulomb's law: electric force $F \sim \frac{q_1 q_2}{r^2}$

$$F = \frac{1}{4\pi\varepsilon_0\varepsilon}\frac{q_1 q_2}{r^2} \qquad [SI]$$

{material dielectric constant ε will be discussed later, for now ε =1 for vacuum}

$$\varepsilon_0 = \frac{10^7}{4\pi c^2} = 8.85\cdot 10^{-12}\frac{F}{m} \qquad \left\{ = \frac{C^2}{Nm^2} \right\}$$

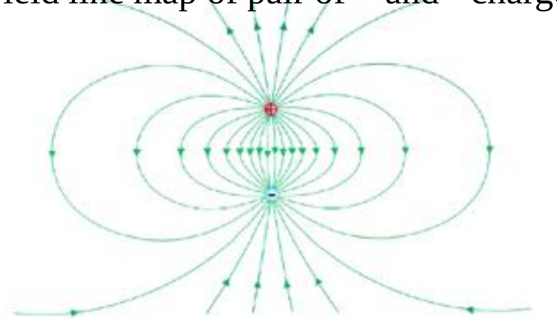
- Electric Field from a charge
  - force that would act on a charge q
  - Linear field

$$\vec{E}(\vec{r}) = \frac{\vec{F}}{q} = \frac{1}{4\pi\varepsilon_0\varepsilon}\frac{q}{r^2} \qquad \left[\frac{V}{m}\right] \qquad \left[=\frac{N}{C}\right]$$

  - Charge motion in the field:

$$\frac{d^2x}{dt^2} = \frac{F}{m} = \frac{q}{m}E$$

Field line map of pair of + and - charges

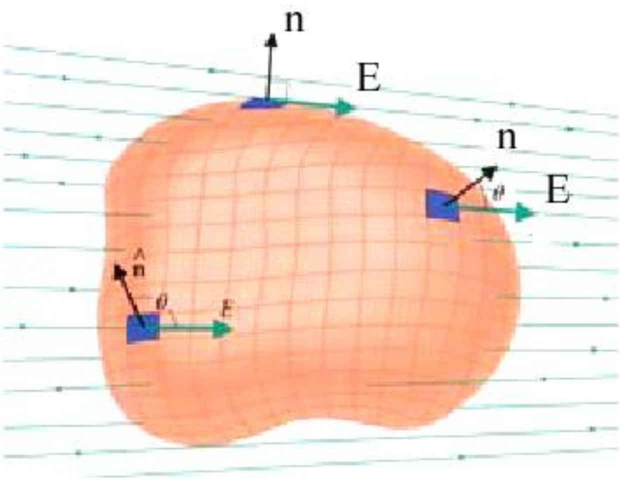


Few concepts from Electrostatics: Gauss' law

- Gauss' law

$$\oint \left(\vec{E}\cdot\vec{n}\right)dS = \frac{Q}{\varepsilon\varepsilon_0}$$

Integration over the closed surface



Note: Coulomb law can be deduced from Gauss law and symmetry considerations

Using divergence theorem (also known as Gauss' theorem in mathematics)

$$\iiint_V div\vec{A}dV = \oiint_S \left(\vec{A}\cdot\vec{n}\right)ds$$

Can derive Gauss' law in in differential form

the "del" or "nabla" operator

$$div\vec{E} \equiv \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon\varepsilon_0}$$

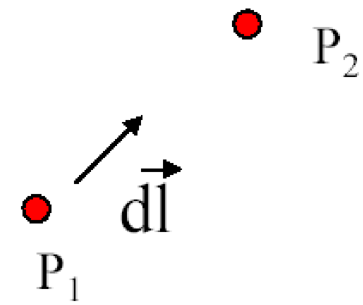
or more accurately:

$$div\left[\underbrace{\varepsilon_0\varepsilon(r)\vec{E}(r)}\right] = \rho(r)$$

In dielectrics, it is useful to introduce electric displacement D(r):

$$div\left[\vec{D}(r)\right] = \rho(r)$$

Few concepts from Electrostatics: Potential



- Potential
  – Change in potential energy equals to work done by the electric field over a charge (electric field is conservative)

$$\varphi_2 - \varphi_1 = \frac{\Delta U}{q} = -\frac{\Delta W}{q} = \frac{-\int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}}{q} = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$

  – If zero potential at infinity: $\varphi = -\int_{\infty}^{P} \vec{E} \cdot d\vec{l}$

  – Or in differential form: $\vec{E} = -\nabla \varphi$

- Poisson equation

$$\nabla^2 \varphi \equiv \Delta \varphi = -\frac{\rho}{\varepsilon \varepsilon_0}$$

- In a medium with no charge density $\rho(r) = 0$

Laplace's equation $\Delta \varphi = 0$

Circulation of Electric field

- Electric field is conservative: work done depends on start and finish of the trajectory. Circulation equals zero.

$$\oint \vec{E} \cdot d\vec{l} = -\varphi_1 + \varphi_1 = 0$$

- Applying Stokes's theorem

$$\oint_C \vec{E} \cdot d\vec{l} = \int_S \nabla \times \vec{E} \cdot d\vec{s}$$

- Because the above must hold for any surface *S*, we must have

$$\boxed{\nabla \times \vec{E} = 0} \quad \equiv \quad curl\vec{E} = 0$$

Few concepts from Electrostatics: Dipole

- Electric dipole
  - Dipole moment

$$p = lq$$

  - In an electric field net force is zero, but torque

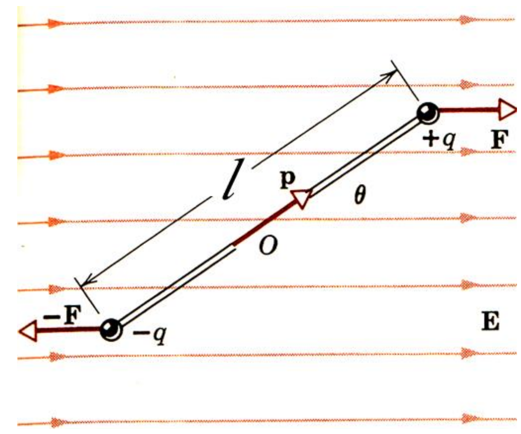
$$\vec{\tau} = \vec{p} \times \vec{E}$$

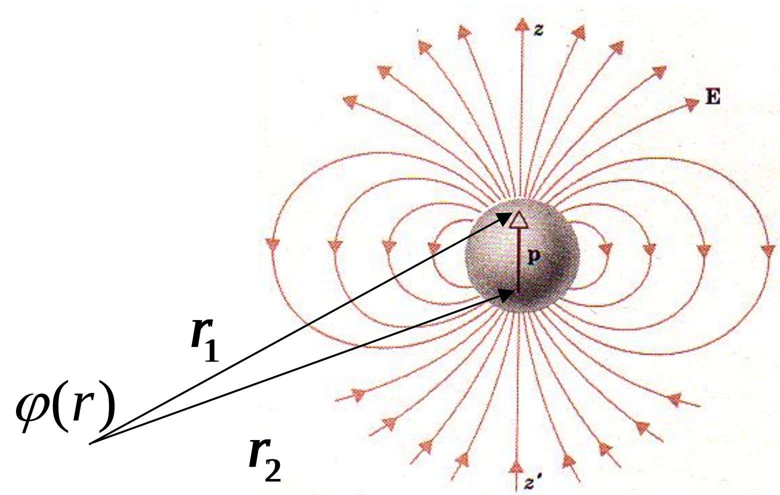
  - Electric field from dipole in vacuum at *r* >>*l*

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(p\cdot r)\vec{r}}{r^5} - \frac{\vec{p}}{r^3}\right]$$

  - Potential due to dipole in vacuum at *r* >>*l*

$$\varphi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \approx \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$





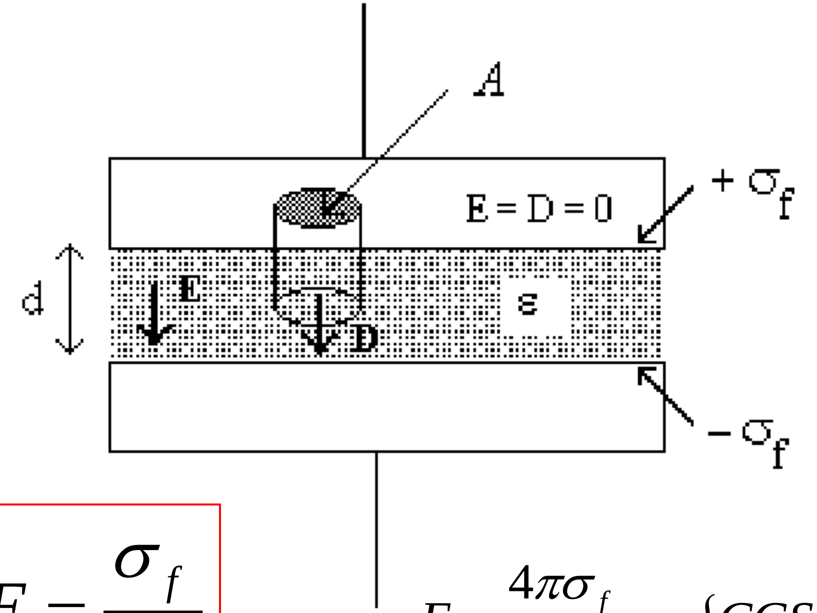
Potential energy of dipole in uniform field varies between:

$$U_{\min} = -\vec{p}\cdot\vec{E} \quad \text{and} \quad U_{\max} = +\vec{p}\cdot\vec{E}$$

Capacitor

- Let's apply the Gauss' law to a cylinder:

$$\oint\left(\vec{E}\bullet\vec{n}\right)dS=\frac{Q}{\varepsilon\varepsilon_0}$$



- The only contribution to the integral is the bottom surface of the cylinder

$$\oint\varepsilon\varepsilon_0 EdS=\varepsilon\varepsilon_0 EA=A\sigma_f$$

- Uniform field between the plates → gives the potential difference (voltage) between the plates

$$E=\frac{\sigma_f}{\varepsilon\varepsilon_0}$$

$$E=\frac{4\pi\sigma_f}{\varepsilon}\quad\{CGS\}$$

$$V=Ed=\frac{\sigma_f d}{\varepsilon\varepsilon_0}=\frac{d}{\varepsilon\varepsilon_0 S}Q\equiv\frac{Q}{C}$$

- With capacitance

$$C=\frac{\varepsilon\varepsilon_0 S}{d}$$

Fundamental Laws of Electrostatics in Differential Form

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{D} = q_{ev}$$

Conservative field

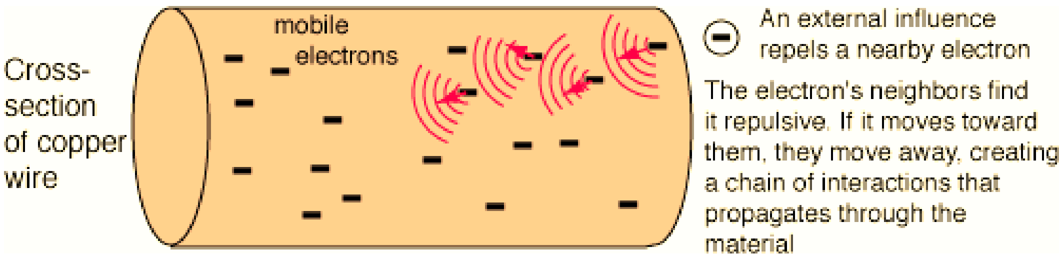
Gauss's law

$$\vec{D} = \varepsilon \vec{E}$$

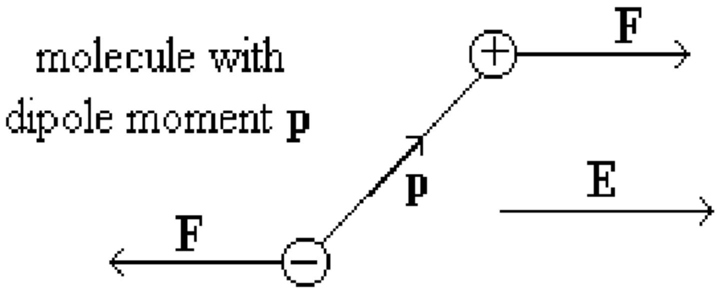
Constitutive relation (we'll discuss it in just a few slides)

Conductors and Insulators at a glance

- Conductors
  - Charges (electrons) are more or less free to move → "free electrons"
  - Electrical properties described by electron motion



- Insulators
  - Mostly no long-range mobility of electrons
  - Electrical properties described by electrical dipoles



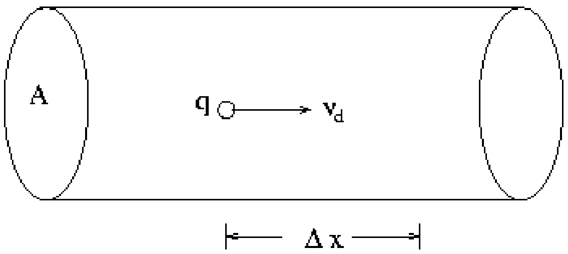
  - Polarization:

$$[P] = \frac{[\text{dipole moment}]}{[\text{volume}]} = \frac{\text{Cm}}{\text{m}^3} = \text{Cm}^{-2}$$

Current: drift velocity

- Electric current = charge flow through a cross-section in a time interval

$$I = \frac{\Delta Q}{\Delta t}, \quad \Delta t \to 0$$



or introducing

n -concentration of electrons,

q -electron charge

v_d – drift velocity

$$I = \frac{\Delta q}{\Delta t} = nAq\frac{\Delta x}{\Delta t} = nAqv_d$$

- Current density

$$J = nqv_d \qquad \left\{\frac{A}{m^2}\right\}$$

- In many materials current is proportional to the applied potential difference: Ohm's law : with specific resistivity, ρ, or conductivity, σ (material parameters)

$$I = \frac{\Delta\varphi}{R}$$

$$R = \rho\frac{\Delta x}{A} \equiv \frac{1}{\sigma}\frac{\Delta x}{A}$$

- It is also useful to introduce local Ohm's law

$$J = \frac{I}{A} = \frac{\Delta\varphi}{R} = \frac{E\Delta x}{R}$$

$$J = \sigma E$$

Current: mobility

- In the Ohm's law regime, the drift velocity should be proportional to the electric field

$$J = nqv_d$$

$$J = \sigma E$$

$$\Downarrow$$

- Similar to viscous hydrodynamic flow with friction force proportional to velocity of an object

$$v_d = \mu E$$

- "Friction" is related to scattering with time τ

$$\mu = \frac{q\tau}{m}$$

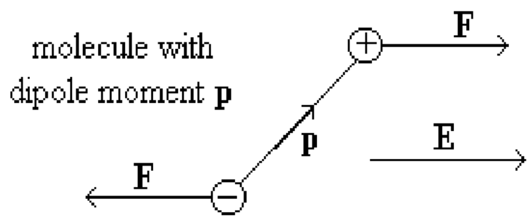
- Proportionality coefficient is called mobility:

$$\mu = \frac{\sigma}{qn} \qquad \left\{ \frac{cm^2}{Vs} \right\}$$

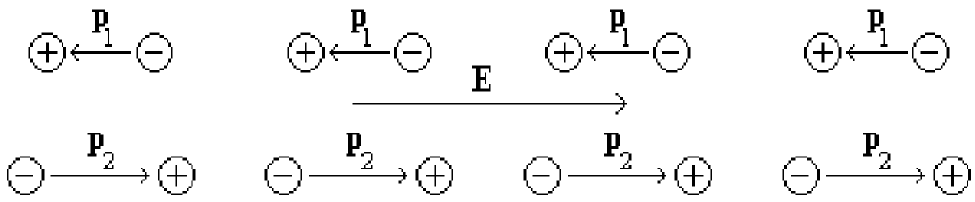
$$\sigma = qn\mu$$

Insulators (dielectrics): Polarisability

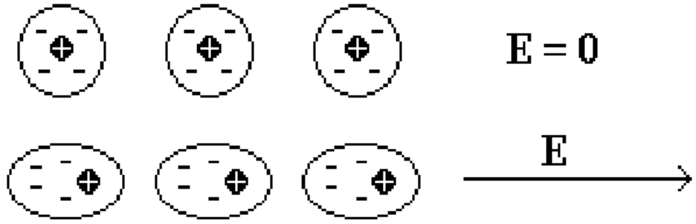
- Three basic mechanisms of polarization:
  - Dipolar (molecular) polarisability due to reorientation (most significant in liquids and gases)



  - Ionic polarisability due to displacements of the positive and negative ions
    - Results in lattice distortions
    - May give rise to ferroelectricity



  - Atomic polarisability due to redistribution of charge in any atom



Polarization

- Assume macroscopic neutrality, but solids are composed of positively and negatively charged entities
- Displacements of charges generate dipole moment and polarization (electric dipole mom per unit volume)

$$[\mathrm{P}] = \frac{[\text{dipole moment}]}{[\text{volume}]} = \frac{\mathrm{Cm}}{\mathrm{m}^3} = \mathrm{Cm}^{-2}$$

- V can be a unit cell volume in crystals in the uniform field

$$P = \frac{\int_V \rho u dv}{V} = \frac{\sum_i n_i q_i u_i}{V}$$

- In general, polarization can be written as a series of the electric field (through susceptibility, χ)

$$P = \varepsilon_0 \left(\chi^{(1)} E + \chi^{(2)} E^2 + \ldots \right)$$

$$P = \chi^{(1)} E + \chi^{(2)} E^2 + .. \quad \{CGS\}$$

- If electric field is much smaller than crystal fields, linear response is good enough :

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

$$\vec{P} = \chi \vec{E} \qquad \{CGS\}$$

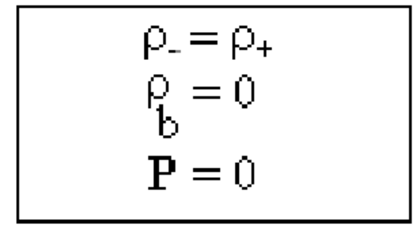
- Otherwise nonlinear terms (2nd and 3rd order susceptibility) are employed → nonlinear optics

Charge and Polarization - I

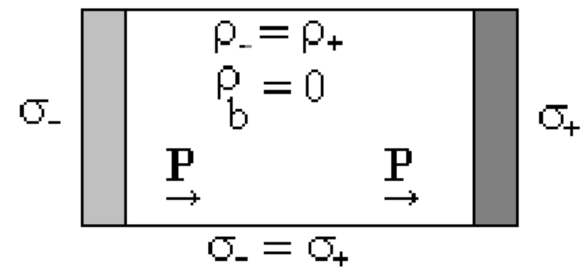
We'll show:

$$\rho_b = -\nabla \vec{P}$$

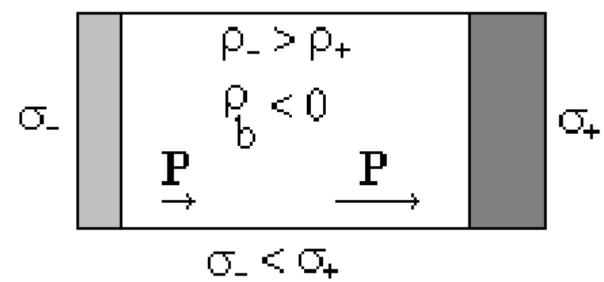
Polarization may be thought of as a bulk movement of the positive charges relative to the negative charges resulting in the bound charge density ρ_b. Consider three cases:



- No polarization. Charge density (ρ_b) in the medium is zero since the positive (ρ_+) and negative (ρ_-) distributic overlap.



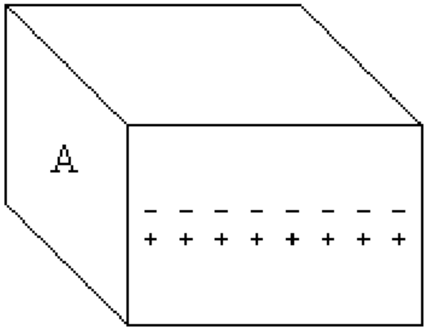
- Uniform polarization. The relative shift of the charge densities leads to the appearance of surface charge densities (σ) The positive and negative charge densitie in the bulk still cancel.



- Nonuniform polarization. The positive charge density is stretched out as well as displaced to the right. The charge density on the positive surface is greater than that on the negative surface. The polarisation increase to the right.

Charge and Polarization - II

Consider a small volume ***within*** the dielectric $\delta V = \delta x \delta y \delta z$

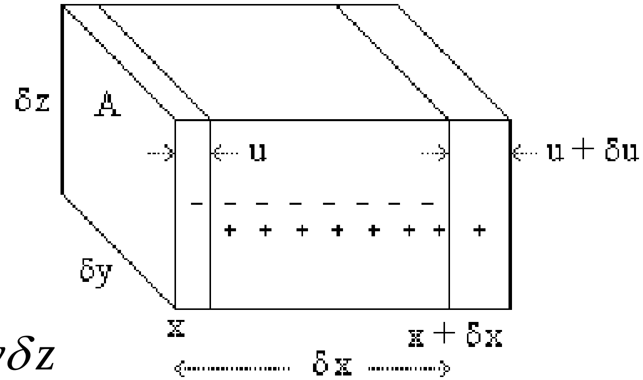


- In the unpolarized dielectric the net charge density is $\rho = \rho_+ + \rho_- = 0$
- If non-uniform displacement is u(x), at the left face it is u(x), whereas at the right face it is u(x + δx)
  - Positive charge enters at the left face:

$$\rho_+(x)u(x)A = P_x(x)A$$

  - Positive charge leaves at the right face:

$$\rho_+(x+\delta x)u(x+\delta x)A = P_x(x+\delta x)A$$



  - The net charge appearing in the volume is no longer zero:

$$Q = \left[P_x(x) - P_x(x+\delta x)\right]A = -\frac{\partial P_x}{\partial x}\delta x \delta y \delta z$$

  - Or Including the other two dimensions, the total charge appearing in δV is

$$Q = -\frac{\partial P_x}{\partial x}\delta x \delta y \delta z - \frac{\partial P_y}{\partial y}\delta x \delta y \delta z - \frac{\partial P_z}{\partial z}\delta x \delta y \delta z$$

  - This gives a *bound* charge density,

$$\rho_b(x,y,z) = \frac{Q}{\delta x \delta y \delta z} = -\frac{\partial P_x}{\partial x} - \frac{\partial P_y}{\partial y} - \frac{\partial P_z}{\partial z} \equiv -div\vec{P}$$

$$\rho_b = -\nabla \vec{P}$$

Dielectric constant

- Polarization contributes an amount $\rho_b(r) = -\nabla \vec{P}(r)$ to the charge density at r:
- We can rewrite Gauss' law in differential form as $\nabla \vec{E} = \frac{(\rho_f + \rho_b)}{\varepsilon_0}$
- This enables us to restate the Gauss' law as: $\nabla\left(\varepsilon_0 \vec{E} + \vec{P}\right) = \rho_f$
- Electric displacement vector, D, and dielectric constant ε can be introduced within linear medium response:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi \vec{E} = (1 + \chi)\varepsilon_0 \vec{E} \equiv \varepsilon\varepsilon_0 \vec{E}$$

$$\vec{D} = \varepsilon \vec{E} \qquad \{CGS\}$$

- In many cases ε is a complex scalar material parameter depending on frequency of electric field. Good news: that's the only parameter (dielectric function) which define electrical and optical properties of a medium

$$\boxed{\nabla\left(\varepsilon\varepsilon_0 \vec{E}\right) = \rho_f}$$

Dielectric properties of some materials

Dielectric	Dielectric constant	Breakdown field , kV/cm
Air	1.0006	30
Glass (pyrex)	5.6	140
PMMA (Plexiglass)	3.4	400
Polystyrene	2.6	250
PTFE (Teflon)	2.1	600
Aluminum oxide	8.4	6700
Silicon	11.9	
Tantalum oxide	26	5000
Barium titanate	~3000	
Water	80	