Lecture contents

- **Macroscopic Electrodynamics**
  - **Propagation of EM Waves in dielectrics and metals**

Maxwell Equations

- Maxwell equations describing the "coupling" of electric and magnetic fields

[SI]

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} = \dfrac{\partial \vec{D}}{\partial t} + \vec{J} \\ \nabla \cdot \vec{D} = q_{ev} \\ \nabla \cdot \vec{B} = 0 \end{cases}$$

[CGS]

$$\begin{cases} \nabla \times E = -\dfrac{1}{c}\dfrac{\partial B}{\partial t} \\ \nabla \times H = \dfrac{1}{c}\dfrac{\partial D}{\partial t} + \dfrac{4\pi}{c} J \\ \nabla \cdot D = 4\pi\rho \\ \nabla \cdot B = 0 \end{cases}$$

- Interaction with medium: in most simple cases linear isotropic :

$$\vec{D} = \varepsilon_R \varepsilon_0 \vec{E} \equiv \varepsilon \vec{E}$$

$$\vec{B} = \mu_R \mu_0 \vec{H} \equiv \mu \vec{H}$$

$$J = \sigma E$$

Further in this section the "total" permeability and permittivity will be used

$$D = \varepsilon E$$

$$B = \mu H$$

$$J = \sigma E$$

  - Polarization is linear function of electric field $\quad P \propto E; \quad D = E + 4\pi P$
  - Interaction is scalar, local and synchronous $\quad D(r,t) = \varepsilon E(r,t)$

Maxwell's Equations for a Isotropic, Linear, Source-Free, and Lossless Medium

- Obviously, there must be a source for the field somewhere.
- However, we are looking at the properties of waves in a region far from the source.

$$\nabla\times\nabla\times\vec{E}=\nabla\left(\cancelto{0}{\nabla\cdot\vec{E}}\right)-\nabla^2\vec{E}$$

$$=-\mu\frac{\partial\left(\nabla\times\vec{H}\right)}{\partial t}=-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

- Wave equations for electromagnetic Waves in a Simple, Source-Free, Lossless Medium
- The wave equations are not independent.
- Usually we solve the electric field wave equation and determine ***H*** from ***E*** using Faraday's law.

$$\nabla\times\nabla\times\vec{H}=\nabla\left(\cancelto{0}{\nabla\cdot\vec{H}}\right)-\nabla^2\vec{H}$$

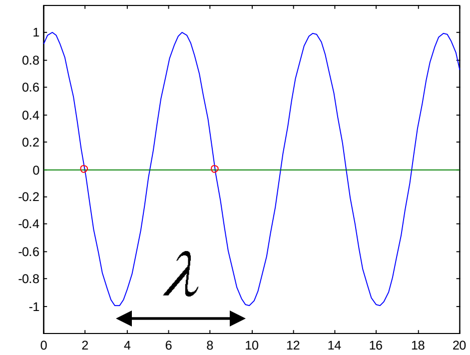
$$=\varepsilon\frac{\partial\left(\nabla\times\vec{E}\right)}{\partial t}=-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

$$\nabla^2\vec{E}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}=0$$

$$\nabla^2\vec{H}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}=0$$

Propagation of electromagnetic waves in an isotropic linear, source-free, and lossless medium

Filed vs. position at a fixed time



- Solution: plane monochromatic wave (equivalent to harmonic analysis)

$$\vec{E} = \vec{E}_0 e^{ikr - i\omega t}$$

- Note: in SI the vacuum constants are chosen: $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$

- With complex wavevector (propagation constant) in lossless medium. Serves as dispersion relation in the medium :

$$k^2 = \mu\varepsilon\omega^2$$

- Or refractive index (in optics):

$$n^2 = \frac{\mu\varepsilon}{\mu_0\varepsilon_0}$$

- The velocity of propagation (phase velocity) is determined solely by the medium:

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$$

- Can also define wavelength $\lambda = \dfrac{v_p}{\omega/2\pi} = \dfrac{2\pi}{k}$
- Strictly speaking, uniform plane waves can be produced only by sources of infinite size.
- However, point sources create spherical waves. Locally, a spherical wave approaches a plane wave.

Medium with low losses

- Plane monochromatic wave : $\vec{E} = \vec{E}_0 e^{ikr - i\omega t}$

- With complex wavevector (propagation constant)
  Serves as dispersion relation in the medium : $$\boxed{k^2 = \tilde{\varepsilon}\mu\omega^2}$$

- It is convenient to introduce dielectric function with real and imaginary parts: $$\tilde{\varepsilon}(\omega) = \varepsilon_1 + i\varepsilon_2$$

- We can introduce the loss tangent of the material : $$\tan\delta = \frac{\varepsilon_2}{\varepsilon_1}$$

- Also complex refractive index (in optics):

$$\tilde{n}(\omega) = n + i\kappa$$

Extinction ratio or extinction index

$$\boxed{k = \frac{\omega}{c}\tilde{n}}$$

Some useful relationships (all material constants depend on frequency !)

- Wavevector can be also expressed through refraction and absorption (extinction) indexes:

$$\begin{cases} n^2 = c^2 \frac{\mu}{2} \ \varepsilon_1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2} \\ \kappa^2 = c^2 \frac{\mu}{2} \ -\varepsilon_1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2} \end{cases}$$

or for $\mu = \mu_0$

$$\begin{cases} \varepsilon_1 = \varepsilon_0 \ n^2 - \kappa^2 \\ \varepsilon_2 = 2n\kappa\varepsilon_0 \end{cases}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

- In this case solution for a plane wave along z-direction is:

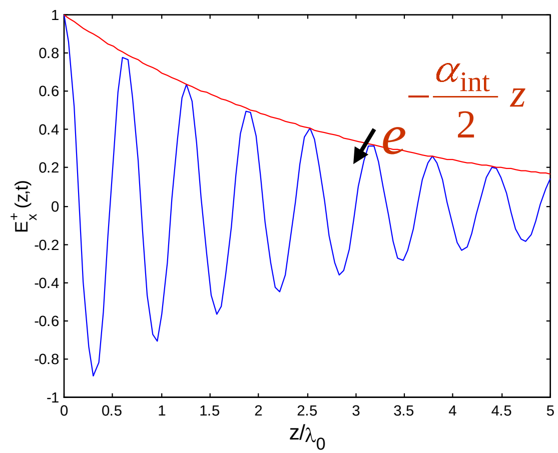
$$E = E_0 e^{-i\omega t - i\frac{\omega n}{c}z - \frac{\omega\kappa}{c}z}$$

(= k-vector: $\frac{\omega n}{c}$; Absorption: $\frac{\omega\kappa}{c}$)

- Where absorption coefficient: $\alpha_{\text{int}} = 2\frac{\omega}{c}\kappa$ is related to intensity reduction: $I = I_0 e^{-\alpha_{\text{int}} x}$
- Or extinction factor: $\kappa = \frac{\lambda \alpha_{\text{int}}}{4\pi}$
- Medium with no absorption: $n = c\sqrt{\mu\varepsilon_1}$ $\quad \kappa = 0$
- and with low absorption: $n = c\sqrt{\mu\varepsilon_1}$ $\quad \kappa = c\sqrt{\frac{\mu\varepsilon_2^2}{4\varepsilon_1}}$

or $\quad \alpha_{\text{int}} = \omega\sqrt{\mu\varepsilon_1} \tan\delta$

Snapshot of $E_x(z,t)$ at $\omega t = 0$

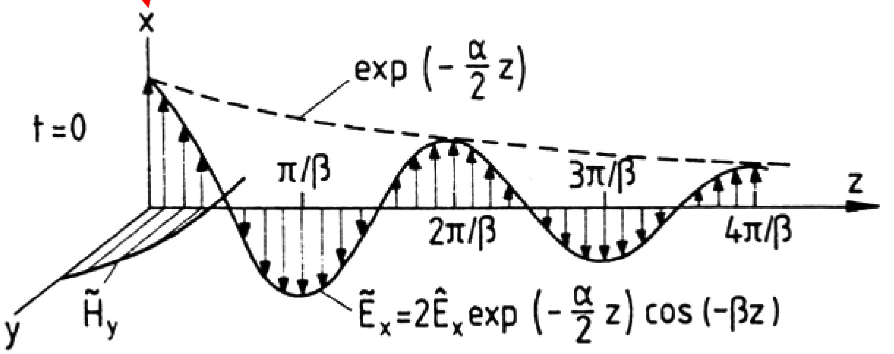


Wave propagation

Polarization

Plane EM wave along z-direction

$$\begin{cases} \vec{E} = \vec{x}E_0 e^{i\omega t} e^{-i\frac{2\pi}{\lambda}nz - \frac{\alpha}{2}z} + c.c. \\ \vec{H} = \vec{y}H_0 e^{i\omega t} e^{-i\frac{2\pi}{\lambda}nz - \frac{\alpha}{2}z} + c.c. \end{cases}$$



Impedance of space:

$$\eta = \frac{E_0}{H_0} = \sqrt{\frac{\mu}{\varepsilon}} = \frac{1}{\tilde{n}}\sqrt{\frac{\mu_0}{\varepsilon_0}} \quad \{\mathrm{Ohm}\}$$

Impedance of vacuum:

$$\eta_0 = \left.\frac{E_0}{H_0}\right|_{vacuum} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega$$

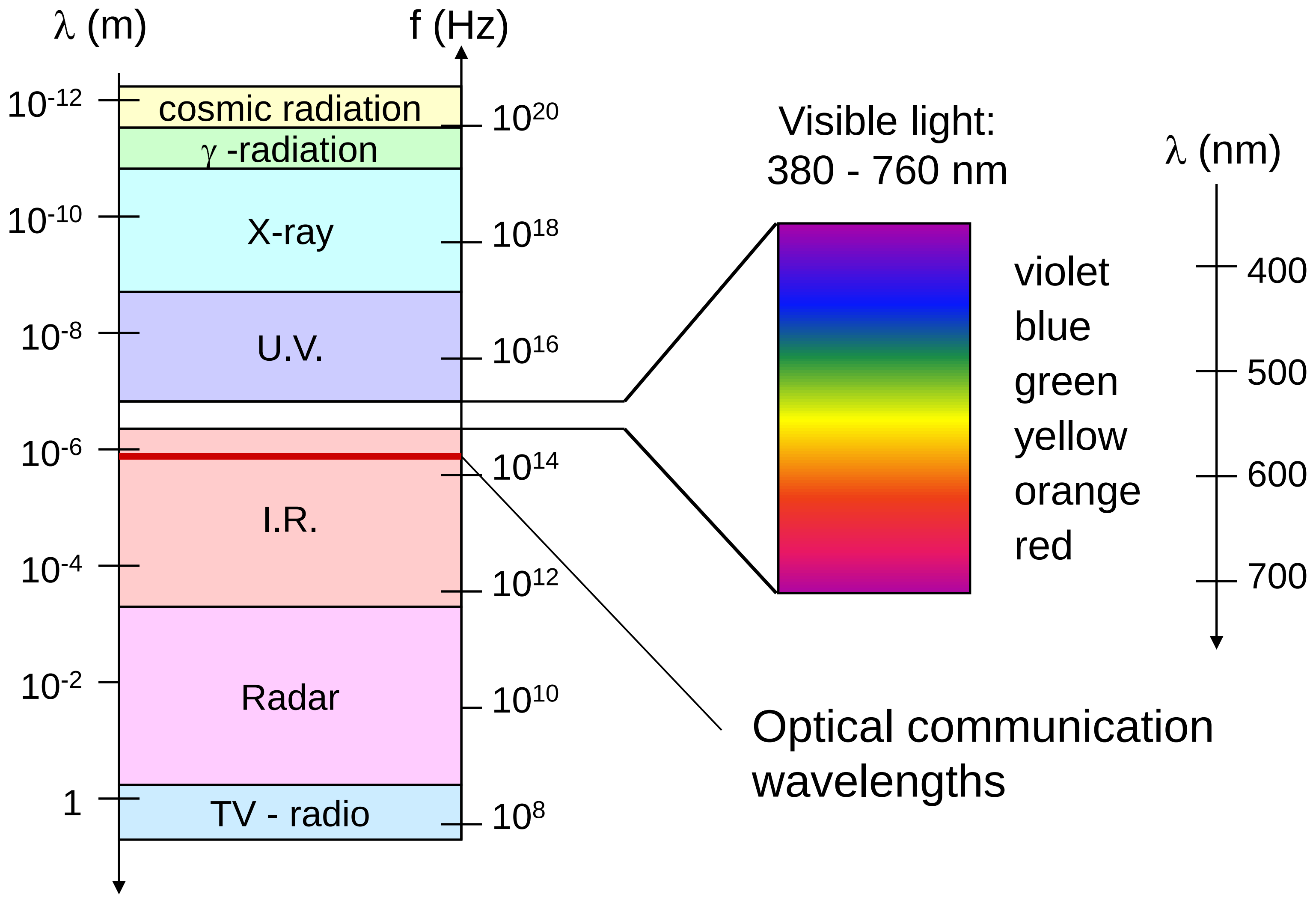
Poynting vector (time-averaged EM power flowing through unit area):

$$\vec{S} = \frac{1}{2}\mathrm{Re}(\vec{E} \times \vec{H}) \quad \left\{\frac{\mathrm{W}}{\mathrm{m}^2}\right\}$$

Intensity:

$$I = \left|\vec{S}\right| = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left|E_0\right|^2 e^{-\alpha_{\mathrm{int}} z}$$

Electromagnetic spectrum



Medium with any losses (all material constants depend on frequency !)

- Maxwell equations in harmonic form in a medium with conduction:

$$\begin{cases} \nabla\times\vec{E} = -i\omega\mu\vec{H} \\ \nabla\times\vec{H} = \vec{J} + i\omega\varepsilon\vec{E} = (\sigma + i\omega\varepsilon)\vec{E} = i\omega\tilde{\varepsilon}\vec{E} \\ \nabla\cdot\vec{E} = 0 \\ \nabla\cdot\vec{H} = 0 \end{cases}$$

$\tilde{\varepsilon} = \varepsilon_1 - i\varepsilon_2$ where $\varepsilon_2 = \dfrac{\sigma}{\omega}$

- A 1D wave equations will look like: $\dfrac{\partial E_x^2}{\partial z^2} - \gamma^2 E_x = 0$ with $\gamma = \alpha + i\beta = \sqrt{-\varepsilon\mu\omega^2\left(1 - i\dfrac{\sigma}{\omega\varepsilon}\right)}$
- It is the same as before with substitution $-\gamma^2 \to k^2$
- The solution also looks the same:

k-vector Absorption

$$\vec{E} = \vec{E}_0 e^{-\gamma r - i\omega t} = \vec{E}_0 e^{i\omega t} e^{-i\beta r} e^{-\alpha r}$$

- with α is the attenuation constant and has units of nepers per meter (Np/m):

$$\alpha = \operatorname{Re}(\gamma) = \operatorname{Re}\left[\sqrt{-\varepsilon\mu\omega^2\left(1 - i\frac{\sigma}{\omega\varepsilon}\right)}\right] \quad \left\{\frac{\text{Np}}{\text{m}}\right\}$$

- β is the phase or propagation constant and has units of radians per meter (rad/m):

$$\beta = \operatorname{Im}(\gamma) = \operatorname{Im}\left[\sqrt{-\varepsilon\mu\omega^2\left(1 - i\frac{\sigma}{\omega\varepsilon}\right)}\right] \quad \left\{\frac{\text{rad}}{\text{m}}\right\}$$

- Note that in general for a lossy medium $\beta \neq \omega\sqrt{\mu\varepsilon}$

Medium with losses

- After some long algebra:

$$\alpha = \omega\sqrt{\varepsilon\mu}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]\right\}^{1/2} \geq 0$$

$$\beta = \omega\sqrt{\varepsilon\mu}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]\right\}^{1/2} > 0$$

- The impedance becomes complex:

$$\eta = \frac{E}{H} = \sqrt{\frac{\mu}{\tilde{\varepsilon}}} = \frac{i\omega\mu}{\gamma} = \frac{\omega\mu}{\alpha^2+\beta^2}\quad \beta + i\alpha$$

For good dielectrics: $\dfrac{\varepsilon_2}{\varepsilon_1} = \dfrac{\sigma}{\omega\varepsilon} \ll 1$

$$\alpha = \frac{\omega}{2}\sqrt{\varepsilon\mu}\,\frac{\sigma}{\omega\varepsilon} \qquad v_p = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\beta = \omega\sqrt{\varepsilon\mu}\left[1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left\{\left[1-\frac{3}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]+i\,\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)\right\}$$

For good conductors: $\dfrac{\varepsilon_2}{\varepsilon_1} = \dfrac{\sigma}{\omega\varepsilon} \gg 1$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$v_p = \sqrt{\frac{2\omega}{\mu\sigma}}$$

$$\eta = \sqrt{\frac{\omega\mu}{2\sigma}} + i\sqrt{\frac{\omega\mu}{2\sigma}}$$